AN EXAMINATION OF SEQUENTIAL PROCEDURES FOR THE TESTING OF THREE HYPOTHESES

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CHAPTER I

INTRODUCTION

Sequential sampling is a significant area of statistics from both theoretical and applied viewpoints. In sequential experimentation, the final sample size is not fixed. Instead, sampling continues until a predetermined level of precision has been attained.

Sequential analysis has two primary branches: sequential hypothesis testing and sequential estimation. In sequential hypothesis testing, two or more hypotheses are tested simultaneously. Sampling continues until one hypothesis is accepted with specified bounds on the error probabilities. Sequential estimation focuses on estimation of one or more parameters with a prespecified level of precision, such as a coefficient of variation at or below a given level or a confidence interval on the parameter with a fixed width. This work will concentrate on sequential hypothesis testing.

Dodge and Romig (1929) were the first to propose a double sampling plan, the most rudimentary sequential procedure. A random sample of fixed size n is taken, and if sufficient evidence is present to arrive at a decision, sampling is terminated. If there is insufficient evidence to arrive at a decision, another random sample of fixed size n_2 is taken. A decision is then made based on the results of the combined samples.

The formal theoretical development of sequential sampling began during World War II with the work of Abraham Wald (1947) and G.A. Barnard (1946) in war-time industrial advisory groups. The most important result was Wald's Sequential Probability Ratio Test (SPRT). The SPRT decides between two simple hypotheses with specified Type I and Type II error rates. Figure 1 is a depiction of the SPRT where X_n is a sufficient statistic (usually the sum of observations).

Consider a test to determine whether θ, a parameter from a distribution $f(x, \theta)$, is equal to θ_0 or θ_1 ($\theta_0 < \theta_1$); that is, H_0: $\theta = \theta_0$ vs. H_1: $\theta = \theta_1$. Further, the Type I error rate, the probability of accepting H_1 when H_0 is true should be at most α, and the probability of accepting H_0 when H_1 is true, or the Type II error rate, is to be at most β. To choose one hypothesis over another based on a sample of fixed size, Neyman and Pearson (1933) developed the likelihood ratio test. They showed that, given a random sample of size n ($x_1, x_2, \ldots, x_n$), the most powerful size - α test depends on the ratio

$$l_n = \prod_{i=1}^{n} \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)}, \tag{1.1}$$

which is the ratio of the joint density of the observed

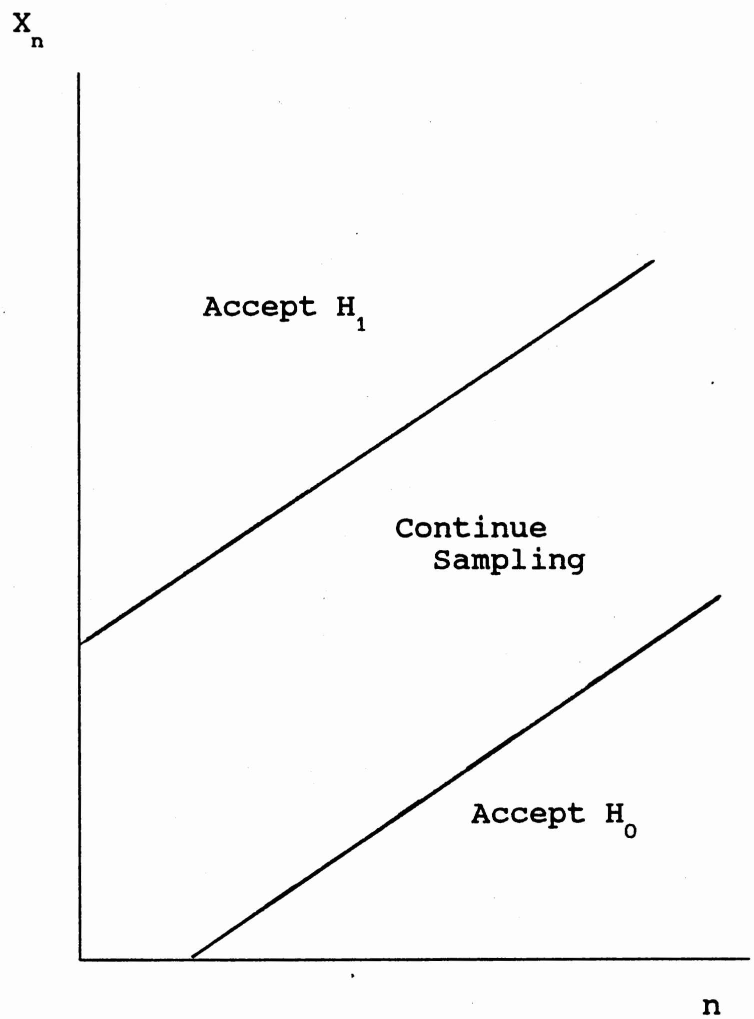

Figure 1. Wald's SPRT for H_0 vs. H_1

random sample given H_1 is true to the joint density of the observed random sample given H_0 is true. The Neyman-Pearson Lemma partitions the parameter space into two regions: one is an acceptance region for H_0 and the other is an acceptance region for H_1 (or a rejection region for H_0). H_0 is accepted if $l_n < c$ and H_1 is accepted if $l_n > c$ for a constant c. The sample size n can be set to obtain the desired Type I and Type II error probabilities.

Wald's SPRT is a sequential analogue to Neyman-Pearson testing. The likelihood ratio, l_n, is computed after each observation. Sampling continues as long as $B < l_n < A$, where A and B are predetermined constants. As soon as $l_n \le B$ (or $l_n \ge A$), H_0 is accepted (or H_1 is accepted). Wald determined that by setting

$$A \simeq \frac{1-\beta}{\alpha}$$

and

$$B \simeq \frac{\beta}{1-\alpha}, \tag{1.2}$$

the probabilities of Type I and Type II errors are approximate the desired levels of α and β, respectively.

The Operating Characteristic (OC) curve is defined as the probability of accepting H_0 given θ and is often denoted by $P(\theta)$. Wald developed the following approximation for the OC curve:

$$P(\theta) \simeq \frac{A^h - 1}{A^h - B^h}, \tag{1.3}$$

where h is a function of θ and the solution of

$$\int_{-\infty}^{\infty} [f(x, \theta_1)/f(x, \theta_0)]^h f(x, \theta)\, dx = 1 \qquad (1.4)$$

and the constants A and B are given in equation (1.2). Wald showed that

$$\alpha' + \beta' \le \alpha + \beta \ , \qquad (1.5)$$

where α' and β' are the actual error rates obtained by the test. The inequality in (1.5) can be replaced by an equality if boundary overshooting is ignored. Seebeck (1989) and Corneliussen and Ladd (1970) showed that the approximate OC curves of certain discrete distributions are good approximations of the actual OC curves.

The Average Sample Number (ASN) function, defined on the entire parameter space as the average number of samples needed to arrive at a decision given θ, is denoted by $E(N|\theta)$. Wald and Wolfowitz (1948) established the following optimality property of the SPRT: For all sequential tests of H_0: $\theta = \theta_0$ vs. H_1: $\theta = \theta_1$ with Type I error probability α and Type II error probability β, the SPRT minimizes $E(N|\theta_0)$ and $E(N|\theta_1)$. Later this result was shown to hold among all tests, sequential or not (Lehmann, 1959). However, since θ may not always assume one of the hypothesized values, the behavior of the ASN over the full parameter space is often of interest.

Wald developed the following approximation for the ASN function of the SPRT:

$$E(N|\theta) \simeq \frac{P(\theta)\ \log B + [1 - P(\theta)]\ \log A}{E(z|\theta)} \qquad (1.6)$$

where A and B are given in equation (1.2) and

$$z_i = \log \frac{f(x_i|\theta_1)}{f(x_i|\theta_0)} \quad .$$

Using the OC function, $P(\theta)$ is the probability of Σz_n being less than or equal to B given θ. The quantity $1 - P(\theta)$ is the probability of Σz_n being greater than or equal to A for a specified value θ. The graph of such an ASN function would take the appearance of that in Figure 2.

Suppose the true parameter θ lies between the hypothesized values; that is, $\theta_0 < \theta < \theta_1$. Often it is assumed that one has no reason to prefer one hypothesis to another under these conditions. Yet, in this region of the parameter space, a larger sample size is required than in any other region. The average sample size for some values in this range will be higher than the corresponding fixed sample test with the same Type I and Type II error rates.

Wald's approximation of the ASN function is exact if no overshooting of the boundaries occurs. However, the

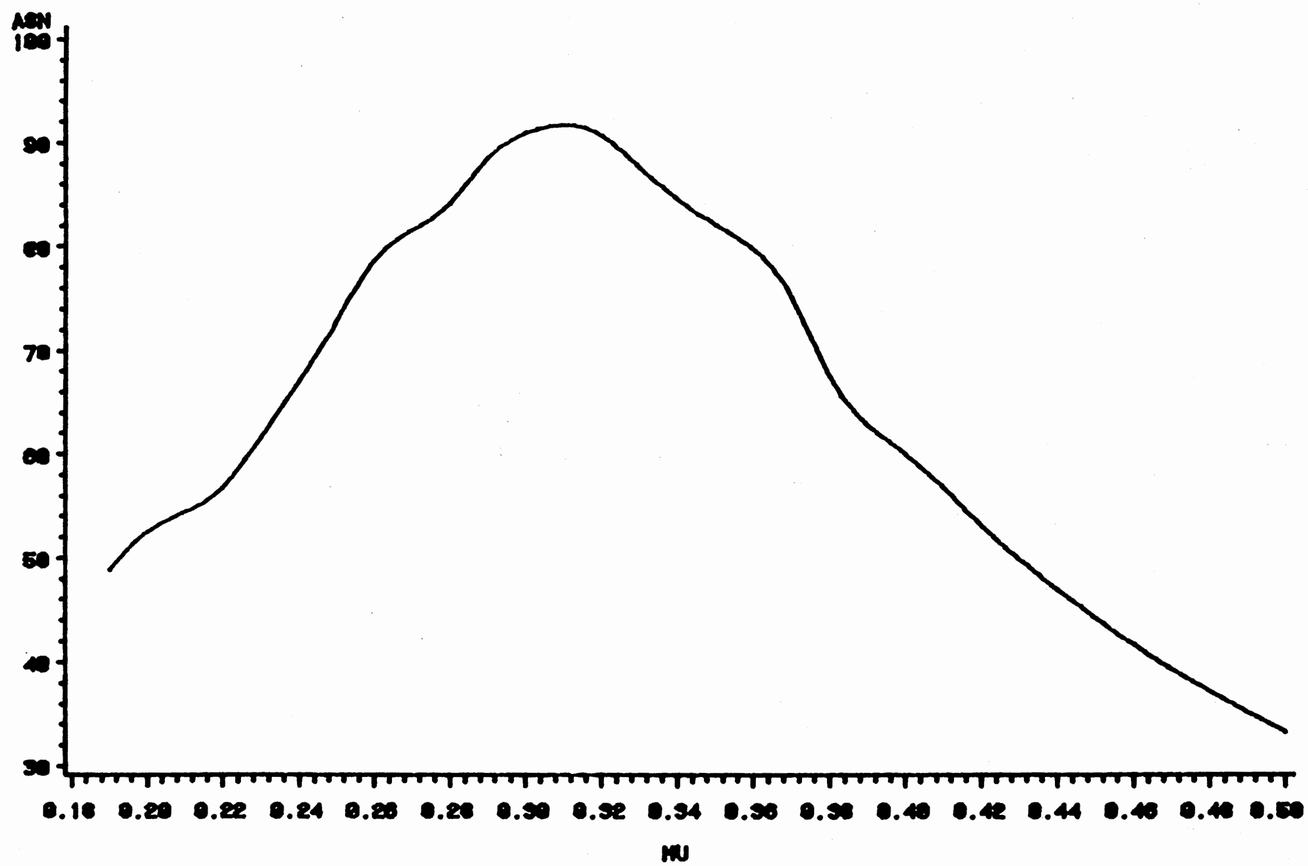

Figure 2. The Average Sample Number Function

overshooting results in substantial underestimation of the ASN, especially for parameter values between the hypothesized ones. Baker (1950) used empirical trials to study a sequential test of the mean of a normal distribution with unit variance. He tested $H_0 : \mu = 0$ vs. $H_1 : \mu = 1$, and found that Wald's approximation underestimated the ASN. Values twice that predicted by Wald's equation were quite common. Studies of the binomial distribution performed by Corneliussen and Ladd (1970) showed Wald's equation under-represents the true value of the ASN by approximately twenty percent at its maximum. They concluded "...in view of the very large spread in sample number, we do not believe the ASN function to be a particularly useful measure of the effort required to conduct a sequential test." They proposed "Termination Probability Contours" as a guide to the amount of sampling required. These are curves that show the number of samples required for certain probabilities (say 0.50, 0.80, 0.95) of termination given values of the binomial parameter p.

The extremely large sample sizes, especially at values intermediate to the hypothesized parameters, limit the utility of Wald's SPRT in some applications. An obvious factor is that the SPRT is an open test; that is, given sampling has not stopped after (n + 1) observations, the probability it will continue after n observations is positive. Thus the parallel boundaries of the SPRT (see Figure 1) may result in unsatisfactorily large sample sizes.

Several methods have been developed to produce closed boundaries such as those in Figure 3. These procedures, called closed sequential tests, place a maximum on the number of observations taken. Chapter II discusses these tests in more detail.

The simultaneous test of three hypotheses, instead of two, is of interest in a number of applications. An entomologist may wish to determine whether a crop is infested heavily, moderately, or lightly with a certain insect (Lye and Story, 1989). This would be an example of a three-hypothesis test involving all simple hypotheses. Billard and Vagholkar (1969) suggested simultaneously testing three hypotheses when performing a two-sided test of hypotheses. As an example, consider a test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$. This can be accomplished by simultaneously testing three simple hypotheses $H_{-1} : \theta = \theta_{-1}$ vs. $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$, where $\theta_{-1} < \theta_0 < \theta_1$. The error rates are controlled at two points under the alternative hypothesis, θ_{-1} and θ_1. The choice of θ_{-1} and θ_1 depends on the precision required for the application.

Open and closed tests of three hypotheses have been developed and are discussed in Chapter II. A computer program was developed to calculate the exact probabilities of error when testing three hypotheses of binomial probabilities. This program was used to study the OC and ASN functions of proposed tests over a range of specified error rates and hypothesized values. The program and

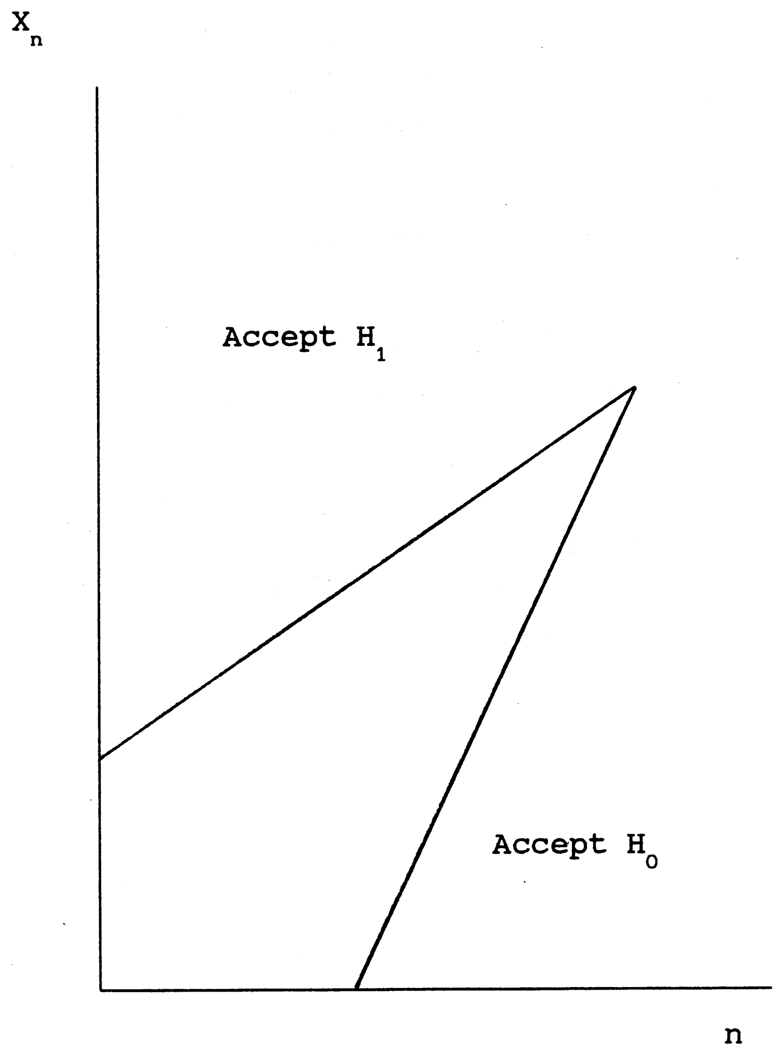

Figure 3. Closed sequential sampling plan

results are presented in Chapter III.

Chapter IV presents a method to test three hypotheses for distributions in the Koopman-Darmois family of densities. It involves approximations of error probabilities. A computer routine is used to solve a system of equations that allows one to set the error rates to any level desired.

In Chapter V, examples are presented for the exponential and normal distributions and simulation studies are performed to evaluate the appropriateness of the method proposed in Chapter IV.

Chapter VI proposes a closed procedure for the three-hypothesis case by extending established closed procedures for testing simple vs. simple hypotheses. Again, the results of simulation studies are presented to check the performance of this closed procedure.

The results presented in this work and possible areas of future research are presented in Chapter VII.

CHAPTER II

REVIEW OF LITERATURE

Koopman (1936) considered the family of densities

$$f(x;\ \theta) = \exp\{k(x) + \theta x - b(\theta)\} \tag{2.1}$$

with respect to some σ-finite measure μ. The function $b(\theta)$ is differentiable such that $b'(\theta) = E_\theta(X)$ and $b''(\theta) = \mathrm{Var}_\theta(X)$. The Kullback-Leibler information number is given by

$$I(\theta,\ \phi) = (\theta - \phi)\ b'(\theta) - (b(\theta) - b(\phi)). \tag{2.2}$$

Schwarz (1962) and Huffman (1983) defined a Koopman-Darmois density as

$$f(x;\ \theta) = \exp\{\theta x - b(\theta)\} \tag{2.3}$$

with respect to some non-degenerate σ-finite measure μ. The properties of (2.1) hold for equation (2.3) This definition was possible since Schwarz's work involving Koopman-Darmois densities was Bayesian in nature. Therefore, he was concerned with distributions of the parameter θ. In this case, the quantity $\exp\{k(x)\}$ is a constant with respect to the parameter. He therefore ignored it in his definition. However, if one wanted to draw inference concerning moments of the random variable X, $\exp\{k(x)\}$ cannot be ignored.

Thus, the more general form for Koopman-Darmois densities given in (2.1) will be used in this work.

$X_n = \sum_{i=1}^{n} x_i$ is complete sufficient for θ in the Koopman-Darmois family of densities. It can easily be shown that the density of X_n is

$$g_n(x_n, \theta) = \exp\{k_n(x_n) + \theta x_n - nb(\theta)\}, \tag{2.4}$$

where $k_n(x_n)$ is a function of x_n that allows g_n to be a probability density function.

The Sequential Probability Ratio Test of two simple hypotheses concerning the Koopman-Darmois parameter θ will be considered in the next section.

Sequential Probability Ratio Test

Consider a test of H_0: $\theta = \theta_0$ vs. H_1: $\theta = \theta_1$, where θ is a parameter from a population with the density $f_\theta(x) = \exp\{k(x) + \theta x - b(\theta)\}$. Further assume the desired Type I and Type II error rates are α and β, respectively. From this population, observations x_1, x_2,... are taken. At the n-th observation, the ratio

$$\frac{f_{1n}}{f_{0n}} = \frac{f_1(x_1)\ f_1(x_2)\ \cdots f_1(x_n)}{f_0(x_1)\ f_0(x_2)\ \cdots f_0(x_n)} \tag{2.5}$$

is considered. The Sequential Probability Ratio Test is as follows:

(a) If $(f_{1n} / f_{0n}) < B$, accept H_0.

(b) If $(f_{1n} / f_{0n}) > A$, accept H_1.

(c) If $B < (f_{1n} / f_{0n}) < A$, continue sampling.

This procedure will extend until either condition (a) or (b) above is satisfied. For the Koopman-Darmois family,

$$\frac{f_{1n}}{f_{on}} = \exp\{(\theta_1 - \theta_0)X_n - n[b(\theta_1) - b(\theta_0)]\}, \tag{2.6}$$

where $X_n = \Sigma x_i$. Wald showed that α and β are attained approximately as error rates if A and B are taken to be

$$A = \frac{1 - \beta}{\alpha} \quad \text{and} \quad B = \frac{\beta}{1 - \alpha}. \tag{2.7}$$

Thus, the sequential procedure becomes:

(a) Accept H_0 if $X_n < \dfrac{n[b(\theta_1) - b(\theta_0)]}{\theta_1 - \theta_0}$

$+ \log [\beta/(1-\alpha)];$

(b) Accept H_1 if $X_n > \dfrac{n[b(\theta_1) - b(\theta_0)]}{\theta_1 - \theta_0}$

$+ \log [(1-\beta)/\alpha];$

(c) Continue sampling if neither (a) nor (b) occurs.

As mentioned in Chapter I, at the hypothesized values, the SPRT has the smallest expected sample size of all procedures with comparable Type I and Type II error rates. However, the sample size may be extremely large when parameter values intermediate to the hypothesized ones occur.

An alternative approach is to minimize the average sample size at an intermediate parameter value θ^*. This problem is known as the modified Kiefer-Weiss problem. Minimizing the ASN at the value of θ^* for which the ASN is a maximum provides a solution to the Kiefer-Weiss problem.

Closed Sequential Procedures

An asymptotic solution to the modified Kiefer-Weiss problem was given by Lorden (1976, 1980). He developed the 2-SPRT test which simultaneously performs two one-sided SPRTs. Consider testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Let θ^* be a value intermediate to θ_0 and θ_1 for which the ASN is to be minimized. Defining a third hypothesis $H_2 : \theta = \theta^*$, a one-sided hypothesis of H_2 against H_0 is conducted for possible rejection of H_0. Simultaneously, another one-sided SPRT of H_2 against H_1 is conducted for the possible rejection of H_1. This results in two converging lines that produce a triangular continuation region (see Figure 3).

Recall

$$f_{in} = f_i(x_1)\cdots f_i(x_n), \quad \text{for } i = 0, 1, 2, \tag{2.8}$$

where f is the density function associated with the population of interest. The 2-SPRT takes the form:

(a) Reject H_0 if $\dfrac{f_{0n}}{f_{2n}} \le A$;

(b) Reject H_1 if $\dfrac{f_{1n}}{f_{2n}} \le B$;

(c) Otherwise, continue sampling. (2.9)

In order to obtain desired error rates α and β, the error rates of the individual SPRTs must be adjusted so that when both are conducted simultaneously, the desired error rates are obtained approximately. The quantities A and B from (2.9) can be determined approximately as:

$$\frac{\alpha}{A} \le P(\text{Accepting } H_1 | H_2 \text{ is true});$$

$$\frac{\beta}{B} \le P(\text{Accepting } H_0 | H_2 \text{ is true}). \tag{2.10}$$

A and B are usually found by solving the inequalities in (2.9) in terms of the sufficient statistic (sum of observations, for instance).

Huffman (1983) extended Lorden's work by determining the value of θ^* which minimizes the maximum sample number to within $o((\log \alpha^{-1})^{1/2})$ as α and β tend to zero. This provides an asymptotic solution to the Kiefer-Weiss problem and will be presented for the Koopman-Darmois family of densities.

Let $x_1, x_2, \ldots$ be a random sample from $f(x, \theta) = \exp\{k(x) + \theta x - b(\theta)\}$. It is desired to test $H_0 : \theta = \theta_0$ vs. $H_2 : \theta = \theta_1$ $(\theta_0 < \theta_1)$ with Type I and Type II error probabilities equal to α and β, respectively. Let

$$X_n = \sum_{i=1}^{n} x_i \ . \tag{2.11}$$

Sampling will continue until

(a) $X_n \geq a_1 n + b_1$ (accept H_1)

or (b) $X_n \geq a_0 n + b_0$ (accept H_0). (2.12)

It remains to determine a_0, a_1, b_0, and b_1. The values a_1 and b_1 are determined by a one-sided test of H_2: $\theta = \theta^*$ vs. $H_1 : \theta = \theta_1$. Likewise, a_0 and b_0 are obtained from a one-sided test of H_0: $\theta = \theta_0$ vs. H_2: $\theta = \theta^*$.

In order to obtain θ^*, and subsequently a_0, a_1, b_0, and b_1, θ' is first determined such that

$$\frac{\log \alpha^{-1}}{I_0(\theta')} = \frac{\log \beta^{-1}}{I_1(\theta')} , \tag{2.13}$$

where $I_i(\theta) = (\theta - \theta_i)b'(\theta) - \{b(\theta) - b(\theta_i)\}$, $i = 0, 1$.

Denote the value of equation (2.13) by n^*. Next compute

$$a_i(\theta') = \frac{(\theta' - \theta_i)}{I_i(\theta')} \quad \text{for} \quad i = 0, 1. \tag{2.14}$$

Denote $a_i(\theta')$ as a_i^*. Find r^* such that

$$\Phi(r^*) = \frac{a_1^*}{a_1^* - a_0^*} \tag{2.15}$$

where Φ is the cumulative distribution function for the standard normal random variable. Also let, for $\theta = \theta^*$,

$$\sigma^* = (\mathrm{Var}_\theta X)^{1/2},$$

where

$$\theta^* = \theta' + \frac{r^*}{\sigma^* (n^*)^{1/2}}.$$

Then the adjusted error rates are

$$\alpha(\theta^*) = \frac{a_0(\theta^*) - a_1(\theta^*)}{a_0(\theta^*)} \alpha$$

and

$$\beta(\theta^*) = \frac{a_1(\theta^*) - a_0(\theta^*)}{a_1(\theta^*)} \beta . \tag{2.16}$$

The values used in Huffman's extension of the 2-SPRT are

$$a_1 = \frac{b(\theta_1) - b(\theta^*)}{\theta_1 - \theta^*},$$

$$a_0 = \frac{b(\theta^*) - b(\theta_0)}{\theta^* - \theta_0},$$

$$b_1 = \frac{\log\left[\dfrac{1-\beta(\theta^*)}{\alpha(\theta^*)}\right]}{\theta_1 - \theta^*},$$

and

$$b_0 = \frac{\log\left[\dfrac{\beta(\theta^*)}{1-\alpha(\theta^*)}\right]}{\theta^* - \theta_0}. \qquad (2.17)$$

Tests of Three Hypotheses Using Sequential Methods

Now consider a sequential test of $H_{-1} : \theta = \theta_{-1}$ vs. $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$. These three simple hypotheses may be a direct consequence of the application, or they may arise due to a desire to decide between a simple hypothesis and a two-sided alternative, $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.

In either case, Armitage (1947) suggested one method of testing of these hypotheses would be to simultaneously conduct two SPRTs. First, construct an SPRT, call it S_{-1}, between H_0 and H_{-1} with

$$P\ (\text{accepting } H_{-1} | H_0 \text{ true}) = \alpha/2$$

and $$P\ (\text{accepting } H_0 | H_{-1} \text{ true}) = \beta.$$

An SPRT S_1 between H_0 and H_1 can also be constructed with

$$P\ (\text{accepting } H_1 | H_0 \text{ true}) = \alpha/2$$

and $$P\ (\text{accepting } H_0 | H_1 \text{ true}) = \beta.$$

The simultaneous tests, S_{-1} and S_1 can be conducted as follows: If S_1 accepts H_1 and S_{-1} accepts H_0, H_1 is accepted in the overall test. If S_1 accepts H_0 and S_{-1} accepts H_0, H_0 is accepted. Finally, if S_1 accepts H_0 and S_{-1} accepts H_{-1}, H_{-1} is accepted. The OC and ASN functions were not derived for this test.

Sobel and Wald (1949) suggested performing S_1 and S_{-1} simultaneously and, more importantly, independently. This means once S_1 decides for H_0, this test terminates, regardless of how S_{-1} performs. Since this test is not simply a function of the sufficient statistics, Sobel and Wald claim this test is not optimum. The advantage is that the independence of the two tests lends nice mathematical properties that allow for approximations of the OC and ASN curves, such as:

$$E(N) \geq \max\{E(N|S_1), E(N(S_{-1}))\},$$

$$P(\text{accepting } H_{-1}) = P(\text{accepting } H_{-1}|S_{-1}),$$

$$P(\text{accepting } H_1) = P(\text{accepting } H_1|S_1),$$

and

$$P(\text{accepting } H_0) = 1 - P(\text{accepting } H_1|S_1) - P(\text{accepting } H_{-1}|S_{-1}).$$

Armitage (1950) suggested simultaneously conducting three SPRTs. In addition to performing S_{-1} and S_1, a test for H_{-1} vs. H_1, call this test S_0, would be run. All three tests would then be performed until one hypothesis is preferred to both of the other hypotheses. This test is usually identical to the dual SPRT method Armitage previously suggested. When examining Figure 4, note that the line AB is in the first quadrant. For this example, the third test S_0 will change this process greatly. However, if AB is not in the first quadrant (this happens whenever the Type I error rate is less than or equal to twice the Type II error rate), S_0 has no effect when added to the dual method. Armitage placed a bound of

$$\text{Prob(error)} \leq \frac{2\alpha}{1 - \alpha} \tag{2.18}$$

for a test with $\alpha = \beta$ for S_{-1}, S_0, and S_1. He claimed, however, that this bound was too wide, and more appropriately,

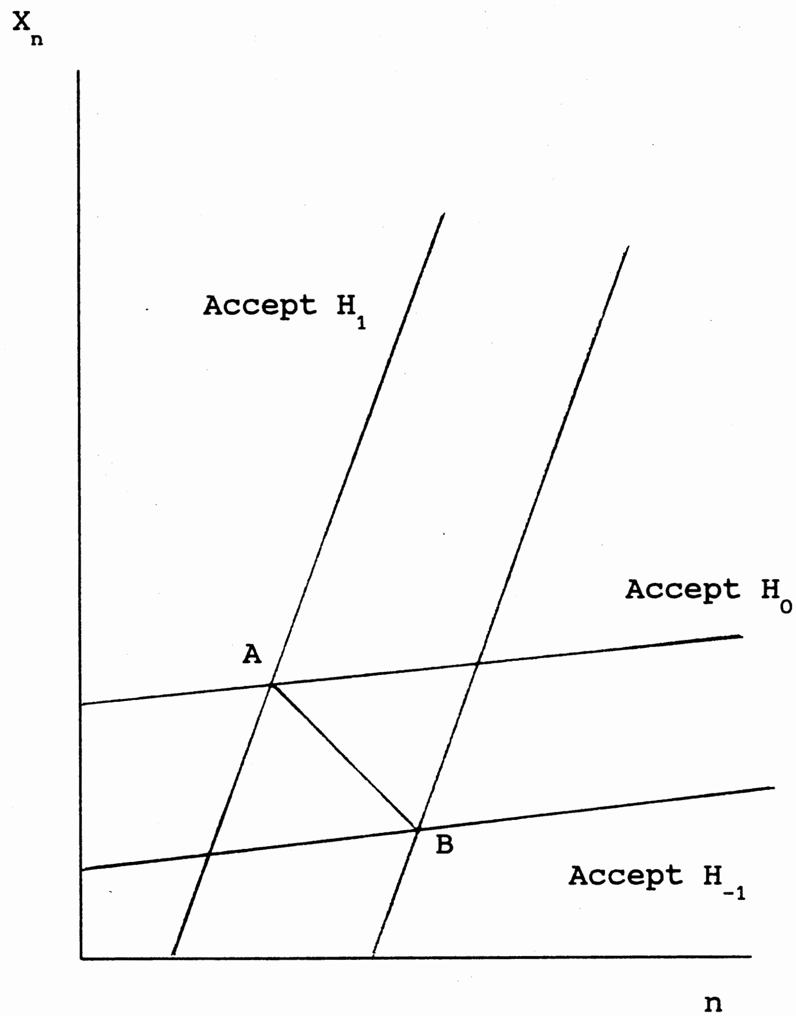

Figure 4. Armitage's (1947) procedure to test three hypotheses

$$P(\text{error}|H_{-1}) = P(\text{error}|H_1) = \alpha$$

and $$P(\text{error}|H_0) = 2\alpha. \tag{2.19}$$

In 1969, Billard and Vagholkar submitted a test that is based on a geometric set of seven test parameters, (a,b,c,d,ψ,ϕ,n_0), (see Figure 5). Based on work by Cox and Miller (1965), they found the probability of a random walk starting at a point x on a line connecting points A and B when $n = n_0$ being absorbed by BP to be

$$\begin{aligned} \Pi(x) &= \frac{\exp(-h_0 a) - \exp(-h_0 x)}{\exp(-h_0 a) - \exp(-h_0 b)} \quad \text{for} \quad h_0 \neq 0 \\ &= \frac{a - x}{a - b} \quad \text{for} \quad h_0 = 0, \end{aligned} \tag{2.20}$$

where h_0 is the solution of

$$E\{\exp(-h(X - \tan \psi))\} = 1. \tag{2.21}$$

Using this, Billard and Vagholkar obtained an expression for the OC function, $L(\mu)$, of their sequential test of the mean of the normal distribution with unit variance, $H_{-1} : \mu = -1$ vs. $H_0 : \mu = 0$ vs. $H_1 : \mu = 1$:

$$L(\mu) = L_0(\mu) + L_1(\mu) + L_{-1}(\mu), \tag{2.22}$$

where L_0, L_1, and L_{-1} are the probabilities of accepting H_0 by crossing lines BC, BP, and CQ, respectively. Now

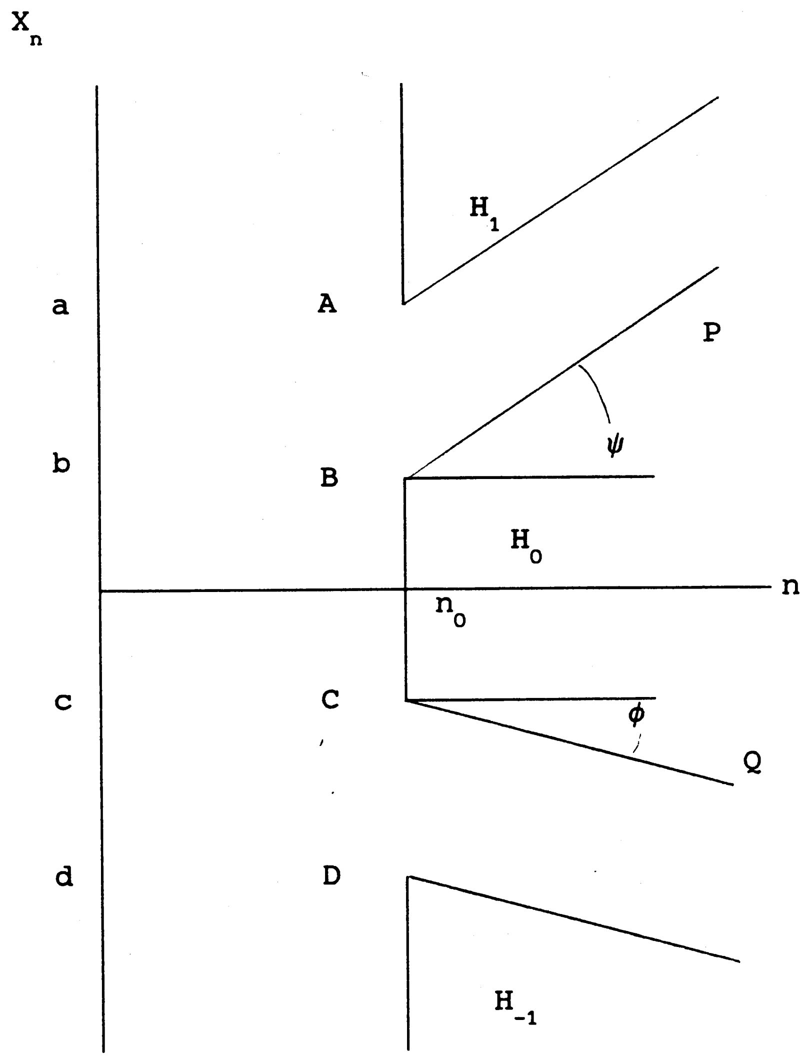

Figure 5. Billard and Vagholkar's (1969) procedure for testing three hypotheses

$$L_0(\mu) = \int_c^b \phi\{(x - n_0\mu)/\sigma n_0^{.5}\}dx \tag{2.23}$$

and
$$L_1(\mu) = \int_b^a \phi\{(x - n_0\mu)/\sigma n_0^{.5}\}\ \Pi(x)dx.$$

The expression for $L_{-1}(\mu)$ is similar to that for $L_1(\mu)$.

Billard and Vagholkar also found an expression for the ASN function; that is,

$$E_\mu(N) = n_0 + N_1(\mu) + N_{-1}(\mu) \tag{2.24}$$

where

$$N_1(\mu) = \int_b^a \phi\{(x - n_0\mu)/\sigma n_0^{.5}\}\ n(x)dx \tag{2.25}$$

and

$$n(x) =$$

$$\left[\frac{(a - b)\exp(-h_0x) - \{a\cdot\exp(-h_0b) - b\cdot\exp(-h_0a)}{\exp(h_0a) - \exp(h_0b)} - x\right]$$

$$\div(\mu\cdot\tan\psi) \quad \text{for} \quad h_0 \neq 0$$

and
$$n(x) = \frac{(a - x)(x - b)}{\sigma^2} \quad \text{for} \quad h_0 = 0. \tag{2.26}$$

Again, a similar expression can be found for $N_{-1}(\mu)$.

Using $L(\mu)$ to set desired error rates, Billard and Vagholkar determined a computer optimization method, developed by Nelder and Mead (1965), that finds the procedure that minimizes $E_{\mu}(N)$ at one of the hypothesized values of μ or an intermediate value.

Billard and Vagholkar originally considered tests for the normal mean. If the test is symmetric and all error rates are equal, these seven parameters (a, b, c, d, ψ, ϕ, n_0) can be reduced to a set of four (a, b, ψ, n_0). Simulation studies were conducted for this special case. Results presented for $H_{-1} : \mu = -1$ vs. $H_0 : \mu = 0$ vs. $H_1 : \mu = 1$ with $\alpha = \beta = 0.05$ indicated the proposed method works well.

Billard (1977a) extended this procedure to include tests of binomial proportions. Examples for the binomial and normal distributions are given in the literature. However, it should be applicable to other distributions.

Closed Procedures for Testing Three Hypotheses

Parameter values intermediate to the hypothesized ones lead to large sample sizes when deciding among three hypotheses as they did in the two-hypothesis case. Therefore, some closed procedures have been developed to test three hypotheses.

One such procedure was developed by Armitage (1957).

His "Restricted Procedure" takes the appearance shown in Figure 6. The test consists of sampling until one of the following conditions is met for $a > 0$, $b > 0$ and $N > 0$:

(a) Accept H_1 if $X_n \geq a + bn$.

(b) Accept H_{-1} if $X_n \leq a - bn$.

(c) Accept H_0 if $n = N$.

The parameters of the test, a, b and N, are determined to attain desired error rates.

Billard (1977b) developed a test that resembles Armitage's Restricted in appearance. In addition, a minimum sample size restriction is used in the test (see Figure 7). Nelder and Mead's minimization procedure is again used to determine the testing parameters (n_0, n_1, a, a′, ψ, ψ'). Arghami and Billard (1982) proposed a procedure that looks somewhat like the 2-SPRT (Figure 8). It used Nelder and Mead to determine the nine testing parameters (n_0, a, b, a′, b′, ϕ_1, ϕ_2, ϕ_1', ϕ_2'). Through symmetry, each of these tests can be simplified to a reduced set of testing parameters. Billard's procedure can be reduced to a set of four parameters, Arghami and Billard's to a set of five.

Related Sequential Work

In 1962, Schwarz explicitly found large-sample limiting shapes of Bayes sequential testing regions. He related his results to the SPRT in the same manner as the likelihood ratio test is related to the Neyman-Pearson test for simple

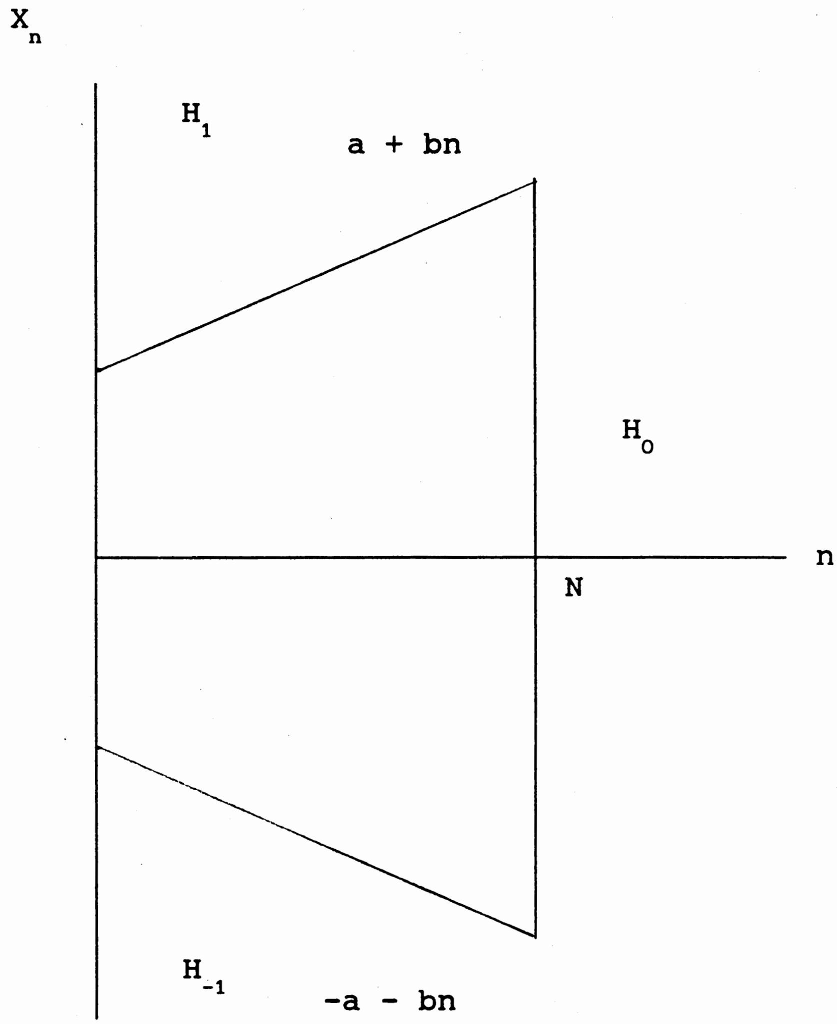

Figure 6. Armitage's Restricted (1957) procedure for testing three hypotheses

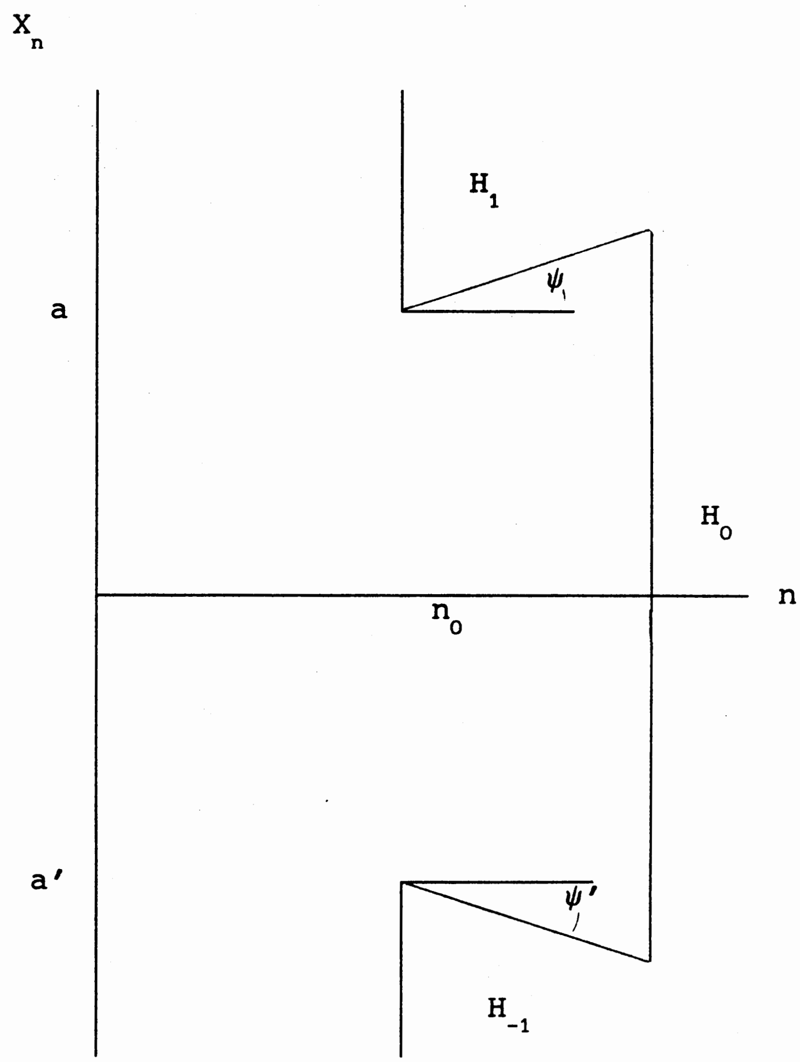

Figure 7. Billard's (1977b) procedure for testing three hypotheses

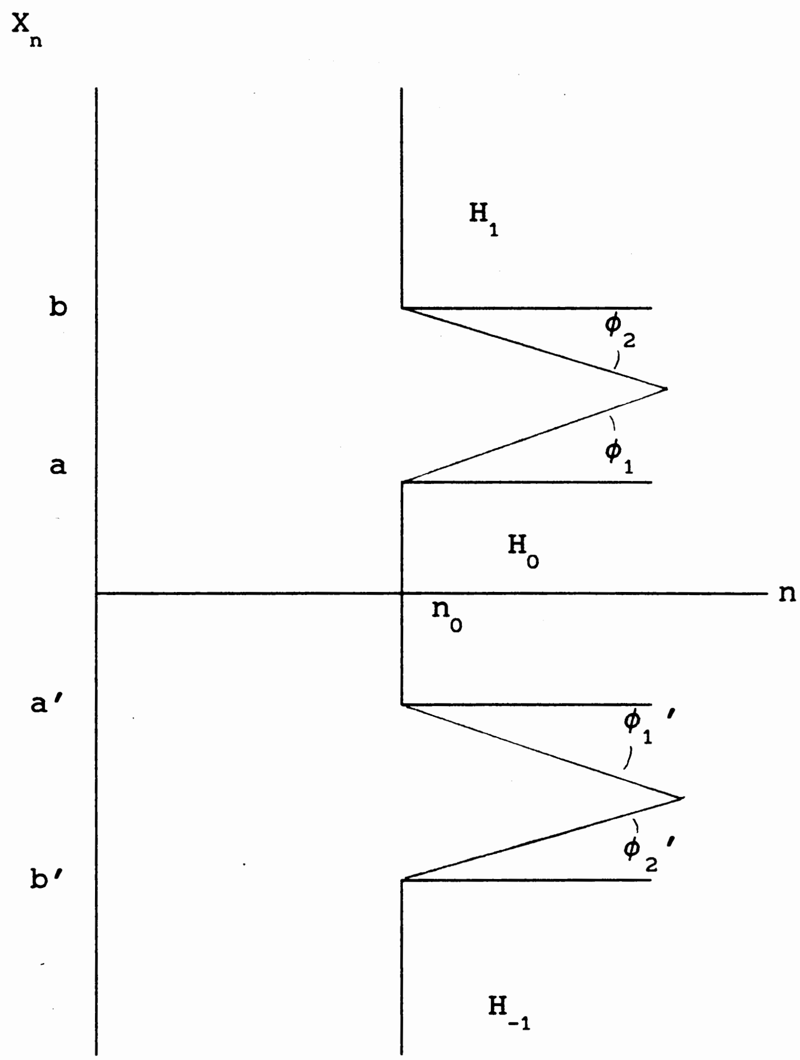

Figure 8. Arghami and Billard's (1982) procedure for testing three hypotheses

ratio test is related to the Neyman-Pearson test for simple hypotheses. Lorden (1972) studied sequential tests with a view to minimizing the ASN at certain values of θ subject to specified error rates, then he considered the extension to k-decision problems.

Corneliussen and Ladd (1970) developed an iterative method to calculate the ASN and OC functions along with the error probabilities for a simple versus simple sequential test of binomial proportions.

The log-likelihood ratio after n observations when sampling from a binomial distribution may be written as

$$\begin{aligned} L(n_1, n_2) &= n_1 \log[(1 - p_1)/(1 - p_0)] \\ &\quad + n_2 \log(p_1/p_0), \end{aligned} \tag{2.27}$$

where n_1 is the number of successes, n_2 is the number of failures and $n_1 + n_2 = n$. For the n-th trial, if x_n is a success, then the log of the likelihood ratio is

$$\begin{aligned} \log (p_{1,n}/p_{0,n}) &= \log (p_{1,n-1}/p_{0,n-1}) \\ &\quad + \log (p_1/p_0). \end{aligned} \tag{2.28}$$

If x_n is a failure, then

$$\begin{aligned} \log (p_{1,n} /p_{0,n}) &= \log (p_{1,n-1} /p_{0,n-1}) \\ &\quad + \log [(1 - p_1)/(1 - p_0)]. \end{aligned} \tag{2.29}$$

overstep a boundary of the sequential test, the test is terminated. For each possible value of $L(n_1, n_2)$ in the continuation region (CR), an associated probability can be calculated using

$$P(n_1, n_2) = I_1(1 - p)\, P(n_1 - 1, n_2) + I_2 p\, P(n_1, n_2 - 1), \qquad (2.30)$$

where

$$\begin{aligned} I_1 &= 1 \quad \text{if} \quad L(n_1 - 1, n_2) \in CR \\ &= 0 \quad \text{if} \quad L(n_1 - 1, n_2) \notin CR \end{aligned}$$

and

$$\begin{aligned} I_2 &= 1 \quad \text{if} \quad L(n_1, n_2 - 1) \in CR \\ &= 0 \quad \text{if} \quad L(n_1, n_2 - 1) \notin CR. \end{aligned}$$

The probability of accepting the respective hypotheses can be calculated iteratively. This method has been extended to the three-hypothesis case and is presented in Chapter III.

Simons (1967) developed a procedure to perform a three-hypothesis test for a mean of a normal distribution with the population variance known. He defined his test, one that takes the appearance of Armitage's (1947) procedure, in terms of six geometrical parameters (γ_1, γ_2, δ_1, δ_2, X, T). These parameters are functions of the error rates of the two SPRT's that are combined to form the test. Adjusting these parameters changes the actual error rates, thus allowing for desired error probabilities to be attained. Ghosh (1970) mentioned that extensions of this method to a nonnormal distribution is "extremely difficult." The possibilities of specifying error rates in this fashion for Koopman-Darmois densities are explored in later chapters.

CHAPTER III

PROGRAM TO CALCULATE PROBABILITIES CONCERNING A SEQUENTIAL TEST OF THREE BINOMIAL PROBABILITIES

Corneliussen and Ladd (1970) derived a method to calculate exact values for the ASN and OC functions for a sequential test concerning binomial proportions, $H_0 : p = p_0$ vs. $H_1 : p = p_1$ (see Chapter II). Similar techniques to determine properties for the sequential test of three binomial hypotheses using Armitage's (1950) method have been developed.

Assume the test of three binomial proportions will be conducted using a graph of X_n by n, the number of trials, as in Figure 9. If the true binomial proportion is p, then at n = 1, the points (1, 1) and (1, 0) will have probabilities p and (1 - p), respectively. At n = 2, the points (2, 2), (2, 1), and (2, 0) will have probabilities p^2, $2p(1-p)$, and $(1-p)^2$, respectively.

This process can be continued so that the probability associated with any point in the continuation region can be calculated. A program was written to perform this analysis. This development permits the evaluation of the exact OC and ASN functions associated with a simultaneous test of three hypotheses concerning binomial proportions. Note that this technique is valid only for discrete distributions. A continuous distribution has an infinite number of possible values in the continuation region.

Consider, for example, the following test:

$$H_1: p = p_{-1} \quad \text{vs.} \quad H_0: p = p_0 \quad \text{vs.} \quad H_1: p = p_1 \tag{3.1}$$

where p is a binomial probability. Armitage (1950) suggested simultaneously conducting three SPRT's. One tests H_{-1} against H_0. The other decides between H_0 and H_1. The SPRT for H_{-1} vs. H_1 will not have any effect on the procedure. Armitage's 1950 procedure will reduce to his 1947 procedure whenever the Type I error rate is less than or equal to twice the Type II error rate. Assuming Type I and Type II error rates of α and β for each of the SPRTs, the slopes of the lines that determine the sampling regions are given by

$$\text{slope}_1 = \frac{\log \dfrac{(1 - p_1)}{(1 - p_0)}}{\log \dfrac{[p_1(1 - p_0)]}{[p_0(1 - p_1)]}}$$

and

$$\text{slope}_0 = \frac{\log \dfrac{(1 - p_0)}{(1 - p_{-1})}}{\log \dfrac{[p_0(1 - p_{-1})]}{[p_{-1}(1 - p_0)]}} . \tag{3.2}$$

Note that these are functions of the hypothesized values only. Formulae for the intercepts of the lines (see Figure 9) are given by:

$$B_L = \frac{\log\left[\dfrac{\beta_0}{1 - \alpha_0}\right]}{\log\left[\dfrac{p_0(1 - p_{-1})}{p_{-1}(1 - p_0)}\right]}, \quad A_L = \frac{\log\left[\dfrac{\beta_1}{1 - \alpha_1}\right]}{\log\left[\dfrac{p_1(1 - p_0)}{p_0(1 - p_1)}\right]},$$

$$B_U = \frac{\log\left[\dfrac{1 - \beta_0}{\alpha_0}\right]}{\log\left[\dfrac{p_0(1 - p_{-1})}{p_{-1}(1 - p_0)}\right]}, \quad A_U = \frac{\log\left[\dfrac{1 - \beta_1}{\alpha_1}\right]}{\log\left[\dfrac{p_1(1 - p_0)}{p_0(1 - p_1)}\right]}. \tag{3.3}$$

These formulae are not only functions of hypothesized values, but also of the probabilities of Type I and Type II errors for each test.

Consider the specific example H_{-1}: p = 0.3 vs. H_0 : p = 0.4 vs. H_1 : p = 0.5 with the desired probability of error 0.10. The α and β for each set of hypotheses could be set to 0.10.

Using the previously mentioned technique to compute exact error probabilities, the following can be calculated:

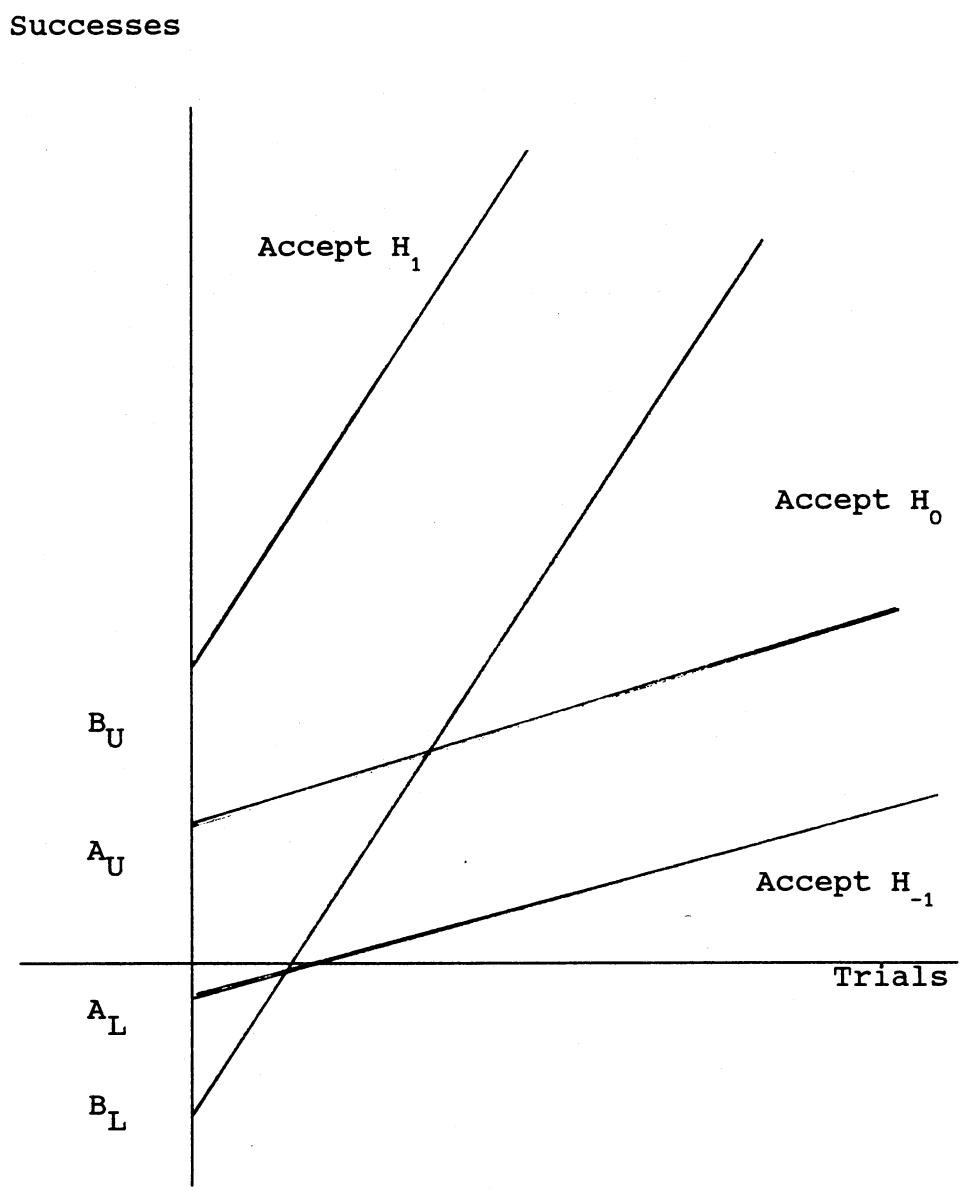

Figure 9. Lines and corresponding intercepts used for determining acceptance regions for the dual SPRT method

$$\text{Prob(error} \mid p = 0.3) = 0.041521,$$

$$\text{Prob(error} \mid p = 0.4) = 0.191853,$$

and $$\text{Prob(error} \mid p = 0.5) = 0.047250.$$

At p = 0.3 or p = 0.5, probabilities of error less than 0.05 are obtained, while at p = 0.4, a probability of error around 0.20 is observed. An analysis using different values of p_{-1}, p_0, and p_1 was performed and Tables I - VI report the results. For most cases presented the following relationships hold:

i) $\text{Prob(error} \mid p = p_{-1}) \le \alpha/2$

ii) $\text{Prob(error} \mid p = p_0) \le \alpha + \beta$

iii) $\text{Prob(error} \mid p = p_1) \le \beta/2$ (3.4)

For certain cases, these relationships do not apply. For instance, for $p_{-1} = 0.25$, $p_0 = 0.50$, $p_1 = 0.75$ and for $p_{-1} = 0.10$, $p_0 = 0.35$, $p_1 = 0.40$, the outer hypotheses have higher error. This seems to indicate that when the hypothesized values are not close (i.e., difference greater than 0.10), the two SPRT's seem to obtain approximate independence. When $p_0 - p_{-1} \neq p_1 - p_0$ (or lack of symmetry), one test seems to dominate the other.

For $p_{-1} = 0.375$, $p_0 = 0.40$, $p_1 = 0.425$, the error probabilities for the outer hypotheses are much smaller than $\alpha/2$ or $\beta/2$. Thus, for hypothesized values very close (differences less than 0.025), the proposed bounds work, but

TABLE I

ERROR PROBABILITIES AT HYPOTHESIZED PARAMETER VALUES FOR TESTING THREE BINOMIAL PROPORTIONS (0.30, 0.40, 0.50)

α	β	p	Prob(error)
0.025	0.025	0.30	0.010849
		0.40	0.047022
		0.50	0.012483
0.025	0.100	0.30	0.012288
		0.40	0.116223
		0.50	0.039683
0.050	0.050	0.30	0.021020
		0.40	0.092405
		0.50	0.023944
0.075	0.075	0.30	0.031425
		0.40	0.139716
		0.50	0.035836
0.100	0.025	0.30	0.035842
		0.40	0.114782
		0.50	0.014315
0.100	0.100	0.30	0.041521
		0.40	0.191853
		0.50	0.047250

TABLE II

ERROR PROBABILITIES AT HYPOTHESIZED PARAMETER VALUES FOR TESTING THREE BINOMIAL PROPORTIONS (0.35, 0.40, 0.45)

α	β	p	Prob(error)
0.050	0.050	0.35	0.015947
		0.40	0.090465
		0.45	0.016831
0.100	0.100	0.35	0.035901
		0.40	0.191492
		0.45	0.038375

TABLE III

ERROR PROBABILITIES AT HYPOTHESIZED PARAMETER VALUES FOR TESTING THREE BINOMIAL PROPORTIONS (0.375, 0.40, 0.425)

α	β	p	Prob(error)
0.050	0.050	0.375	0.000098
		0.40	0.092378
		0.425	0.000118
0.100	0.100	0.375	0.001085
		0.40	0.091145
		0.425	0.001248

TABLE IV

ERROR PROBABILITIES AT HYPOTHESIZED PARAMETER VALUES FOR TESTING THREE BINOMIAL PROPORTIONS (0.10, 0.30, 0.40)

α	β	p	Prob(error)
0.050	0.050	0.10	0.000099
		0.30	0.023780
		0.40	0.000118
0.100	0.100	0.10	0.001085
		0.30	0.091145
		0.40	0.001248

TABLE V

ERROR PROBABILITIES AT HYPOTHESIZED PARAMETER VALUES FOR TESTING THREE BINOMIAL PROPORTIONS (0.10, 0.35, 0.40)

α	β	p	Prob(error)
0.050	0.050	0.10	0.000318
		0.35	0.044932
		0.40	0.043586
0.100	0.100	0.10	0.002809
		0.35	0.099663
		0.40	0.093777

TABLE VI

ERROR PROBABILITIES AT HYPOTHESIZED PARAMETER VALUES FOR TESTING THREE BINOMIAL PROPORTIONS (0.25, 0.50, 0.75)

α	β	p	Prob(error)
0.050	0.050	0.25	0.040074
		0.50	0.089437
		0.75	0.040075
0.100	0.100	0.25	0.073529
		0.50	0.187321
		0.75	0.073769

are extremely conservative. In general, as the differences in hypothesized values decrease, the error probabilities also decrease for the outer hypotheses.

As mentioned in Chapter II, Armitage (1950) suggested the following bound for errors assuming $\alpha = \beta$ in both tests.

$$P(\text{error}|H_i) < \alpha \text{ for } i = -1, 1$$

and $$P(\text{error}|H_0) < 2\alpha. \tag{3.5}$$

Note that for $\alpha = \beta$, the exact computations have good agreement with Armitage's bound for H_0. However, the error bounds for the outer hypotheses are extremely conservative.

The Exact ASN

The technique used in calculating the exact OC function can be used to calculate the exact Average Sample Number for given values of p. Table VII gives the values of the ASN functions for H_{-1} : p = 0.3, H_0 : p = 0.4, and H_1 : p = 0.5 ($\alpha = \beta = 0.10$). Figure 10 is the corresponding graph.

Notice that the graph of the ASN has two peaks indicative of a simultaneous test of three hypotheses. The peaks occur at values between the adjacent hypothesized values.

Alternative Method of Boundary Selection

Due to the fact that Armitage placed bounds on the error probabilities as mentioned previously in this chapter,

TABLE VII

AVERAGE SAMPLE NUMBER VALUES FOR PARAMETER VALUES FOR TESTING THREE BINOMIAL PROPORTIONS (0.30, 0.40, 0.50) ($\alpha = \beta = 0.10$)

p	ASN	p	ASN
0.000	15.0000	0.500	88.5579
0.025	16.1379	0.525	68.2784
0.050	17.3843	0.550	54.0547
0.075	18.8499	0.575	44.3321
0.100	20.6443	0.600	37.4351
0.125	22.8904	0.625	32.3014
0.150	25.7488	0.650	28.3421
0.175	29.4694	0.675	25.2154
0.200	34.5087	0.700	22.6979
0.225	41.6961	0.725	20.6326
0.250	52.2560	0.750	18.9092
0.275	67.4972	0.775	17.4477
0.300	87.9762	0.800	16.1902
0.325	110.0274	0.825	15.0927
0.350	121.2751	0.850	14.1218
0.375	116.8491	0.875	13.2524
0.400	113.7908	0.900	12.4672
0.425	121.8931	0.925	11.7550
0.450	126.5710	0.950	11.1093
0.475	112.3164	0.975	10.5262
		1.000	10.0000

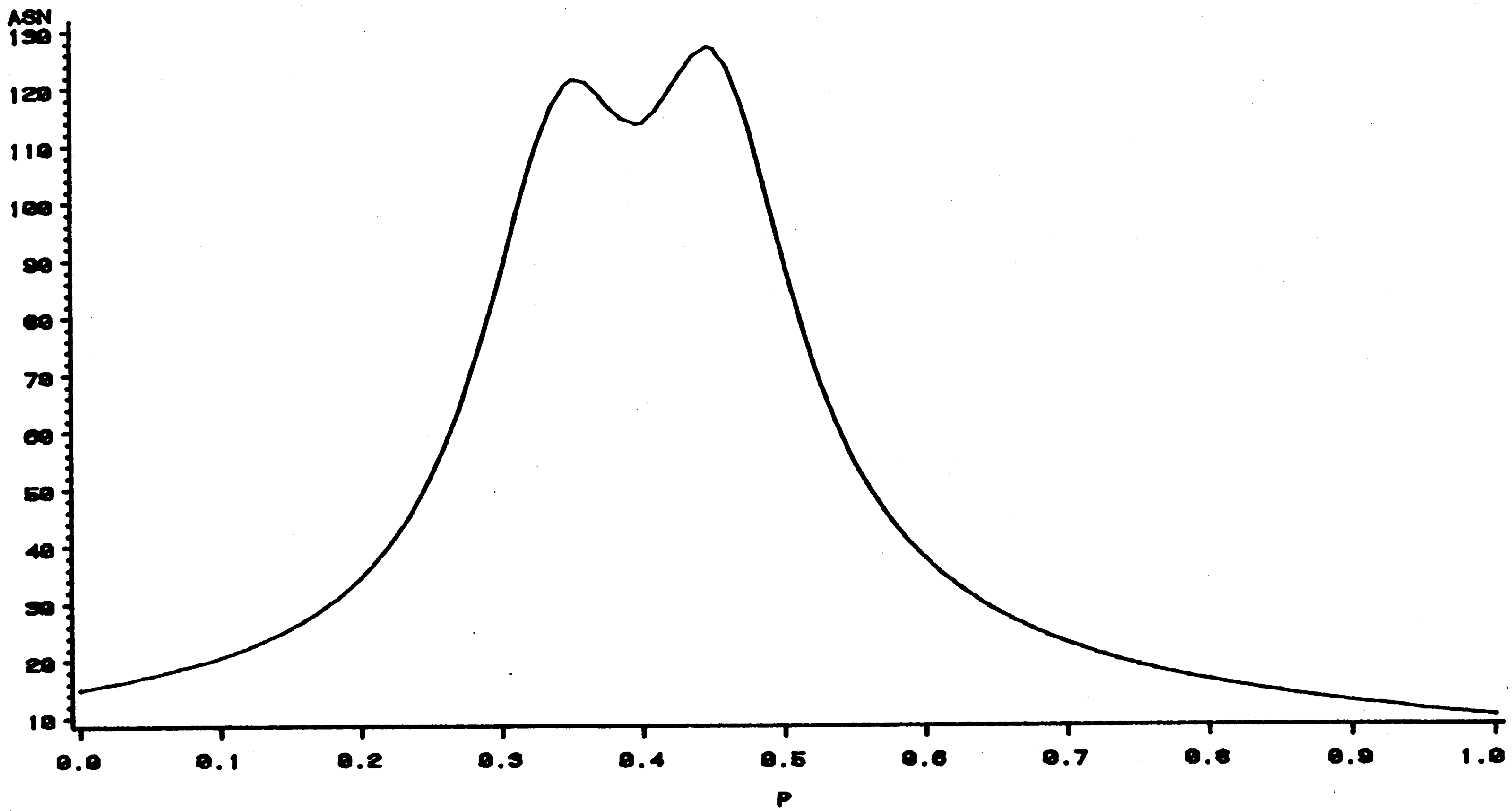

Figure 10. Average Sample Number Function for the Test of Three Binomial Proportions, 0.3, 0.4, and 0.5 with Alpha = Beta = 0.10

he suggested an adjustment of the α's and β's that appear in the equations determining intercepts. Recall that the probability of error given $p = p_0$ is approximately $\alpha + \beta$. Thus α and β can be replaced with $\alpha/2$ and $\beta/2$ in the equations for the intercepts for the acceptance of $H_{-1} : p = p_{-1}$ and $H_1 : p = p_1$. This allows for the acceptance of H_{-1} and H_1 to occur less frequently, thus lowering the probability of making an error when $p = p_0$. The exact computations of the OC function suggests a further adjustment. Recall the error rates at $p = p_{-1}$ and $p = p_1$ are approximately $\alpha/2$ and $\beta/2$, respectively. To accommodate this, α and β in the two remaining intercepts may be replaced with 2α and 2β. This will result in the more frequent acceptance of H_0, thus producing larger error rates for H_{-1} and H_1. The intercepts will then be:

$$B'_L = \frac{\log\left[\dfrac{2\beta}{1-\alpha}\right]}{\log\left[\dfrac{P_0(1-P_{-1})}{P_{-1}(1-P_0)}\right]}, \qquad A'_L = \frac{\log\left[\dfrac{\beta/2}{1-\alpha/2}\right]}{\log\left[\dfrac{P_1(1-P_0)}{P_0(1-P_1)}\right]},$$

$$B'_U = \frac{\log\left[\dfrac{1-\beta/2}{\alpha/2}\right]}{\log\left[\dfrac{P_0(1-P_{-1})}{P_{-1}(1-P_0)}\right]}, \qquad A'_U = \frac{\log\left[\dfrac{1-\beta}{2\alpha}\right]}{\log\left[\dfrac{P_1(1-P_0)}{P_0(1-P_1)}\right]}. \tag{3.6}$$

An analysis using $\alpha = \beta$ shows that these intercepts actually improve the error rates in the sense that they are closer to the desired level of error (see Table VIII).

ASN for Alternative Method

The ASN curve for this alternative method compares favorably to Armitage's 1950 method. The ASN is larger than Armitage's for values of p less that p_{-1}. When $p_{-1} < p < p_1$, the ASN for this proposed method is smaller than Armitage's method.

Figure 11 and Table IX represent the ASN for H_{-1}: p = 0.3 vs. H_0 : p = 0.4 vs. H_1 : p = 0.5 using desired probability of error equal to 0.10. It is compared to the ASN for the same test using $\alpha = \beta = 0.10$. An examination such as this reveals some important characteristics of sequential sampling. The actual probability of error is a function of the hypothesized values and the desired error rates. The method presented to adjust α and β has some limitations. Changing the hypothesized values will cause the adjustment to be ineffective in certain cases. One set of Type I and Type II errors can be appropriate for one test and not for another. The question then stands: "Can one adjust the error rates so that the test of three hypotheses will attain desired probabilities of error?" This question is addressed in the following chapter.

TABLE VIII

ERROR PROBABILITIES AT HYPOTHESIZED PARAMETER VALUES USING ALTERNATIVE METHOD OF BOUNDARY SELECTION

$p_{-1} = 0.3 \quad p_0 = 0.4 \quad p_1 = 0.5$

Desired Prob(error)	p	Prob(error)
0.05	0.30	0.047889
	0.40	0.042681
	0.50	0.052752
0.10	0.30	0.096122
	0.40	0.077682
	0.50	0.104966

$p_{-1} = 0.1 \quad p_0 = 0.3 \quad p_1 = 0.4$

Desired Prob(error)	p	Prob(error)
0.05	0.10	0.022656
	0.30	0.042327
	0.40	0.087705

$p_{-1} = 0.25 \quad p_0 = 0.5 \quad p_1 = 0.75$

Desired Prob(error)	p	Prob(error)
0.05	0.25	0.082552
	0.50	0.038778
	0.75	0.082532

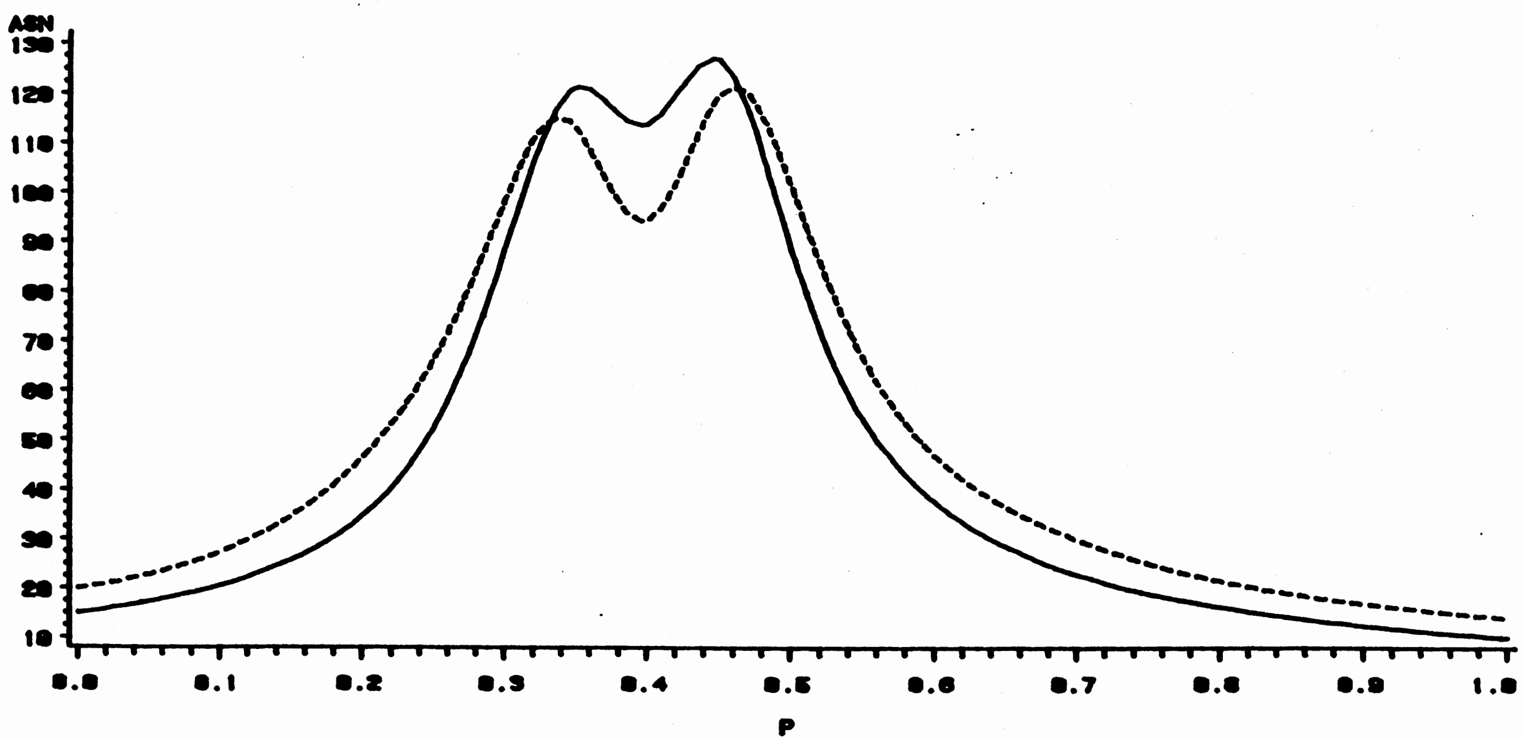

Figure 11. Comparison of ASN Functions for Alternative Method (---) and Armitage (——) for the Test of Three Proportions (0.3, 0.4, 0.5)

TABLE IX

AVERAGE SAMPLE NUMBER VALUES FOR PARAMETER VALUES FOR THE MODIFIED TEST OF THREE BINOMIAL PROPORTIONS (0.30, 0.40, 0.50) (ERROR = 0.10)

p	ASN	p	ASN
0.000	20.0000	0.500	101.0775
0.025	21.1919	0.525	82.0227
0.050	22.7742	0.550	66.4630
0.075	24.7825	0.575	54.9752
0.100	27.2972	0.600	46.7616
0.125	30.4688	0.625	40.8466
0.150	34.5360	0.650	36.3928
0.175	39.7597	0.675	32.8201
0.200	46.4137	0.700	29.8145
0.225	54.9306	0.725	27.2402
0.250	66.0520	0.750	25.0332
0.275	80.6249	0.775	23.1427
0.300	98.1997	0.800	21.5172
0.325	112.9311	0.825	20.1091
0.350	112.8903	0.850	18.8845
0.375	99.9851	0.875	17.8165
0.400	94.3144	0.900	16.8852
0.425	105.1230	0.925	16.0706
0.450	119.7423	0.950	15.3463
0.475	117.9234	0.975	14.6747
		1.000	14.0000

CHAPTER IV

A PROCEDURE TO SEQUENTIALLY TEST THREE HYPOTHESES

Consider the probability density function from the Koopman-Darmois family of densities; that is,

$$f_\theta(x) = \exp\{k(x) + \theta x - b(\theta)\}. \tag{4.1}$$

Suppose it is of interest to choose between the three hypotheses

$$H_{-1} : \theta = \theta_{-1} ,$$
$$H_0 : \theta = \theta_0 ,$$
and
$$H_1 : \theta = \theta_1 .$$

Let x_i, $i = 1, \ldots, n, \ldots,$ be independent observations from f. $X_n = \sum_{i=1}^{n} x_i$, the sufficient statistic for θ, will be the test statistic. Figure 12 takes the appearance of Billard and Vagholkar's (1969) procedure (see Figure 5 for comparison). Figure 12 can be obtained by letting b = c in Figure 5. If, in the process of sampling, (n, X_n) lies above the region determined by AL, sampling is discontinued and H_1 is accepted. Likewise, if (n, X_n) lies in the region determined by MCP, H_0 is accepted. If (n, X_n) lies above

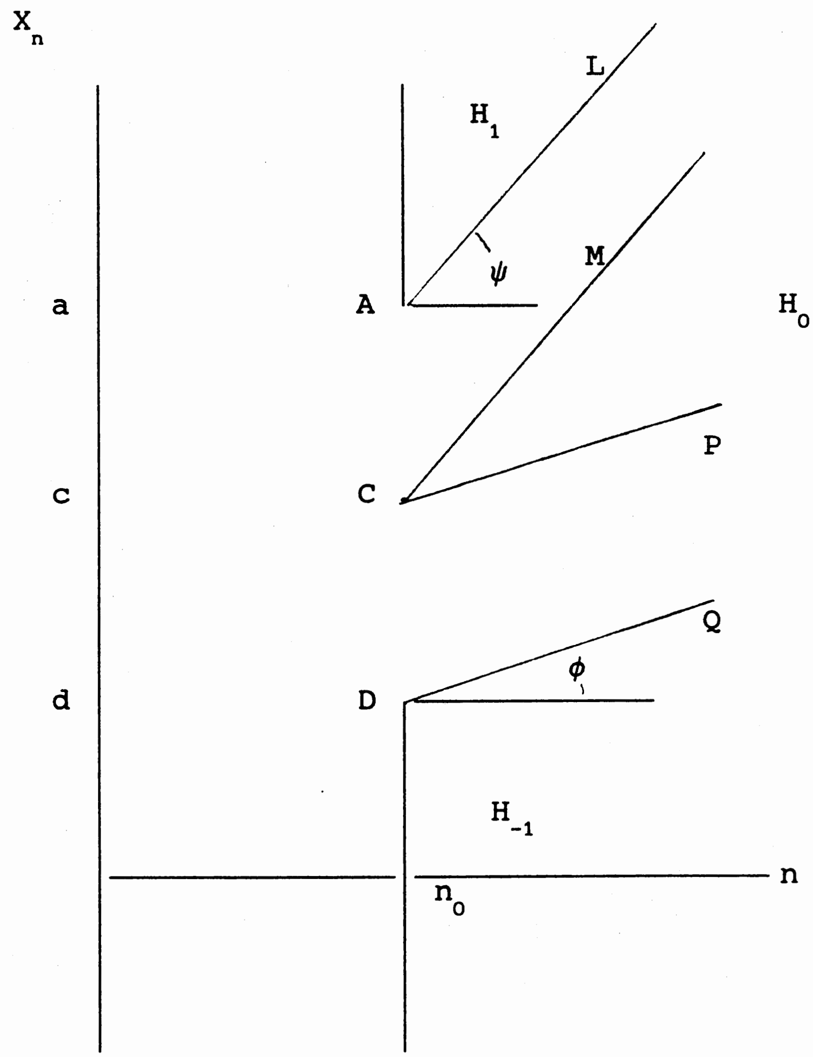

Figure 12. Sampling region for testing three hypotheses

DQ, H_{-1} is accepted. Otherwise, sampling continues. The parallel boundaries can be established by performing two SPRTs simultaneously for H_{-1} vs. H_0 and H_0 vs. H_1. The value of n at which the two SPRTs cease intersecting (or where the "accept H_0" wedge begins) is n_0, which serves as a sample size minimum.

Let α_i, β_i be the Type I and Type II error rates, respectively, for testing H_i vs. H_{i+1}, i = -1, 0. Using SPRTs as defined in Chapter II, it is possible to define the process as follows.

For $n \geq n_0$, then

$$\text{accept } H_1 \text{ if } X_n \geq \frac{[b(\theta_1) - b(\theta_0)]}{(\theta_1 - \theta_0)} n + \frac{\log[(1-\beta_1)/\alpha_1]}{(\theta_1 - \theta_0)},$$

$$\text{accept } H_0 \text{ if } \quad \frac{[b(\theta_0) - b(\theta_{-1})]}{(\theta_0 - \theta_{-1})} n + \frac{\log[(1-\beta_0)/\alpha_0]}{(\theta_0 - \theta_{-1})}$$

$$\leq X_n \leq \frac{[b(\theta_1) - b(\theta_0)]}{(\theta_1 - \theta_0)} n + \frac{\log[\beta_1/(1-\alpha_1)]}{(\theta_1 - \theta_0)},$$

or

$$\text{accept } H_{-1} \text{ if } X_n \leq \frac{[b(\theta_0) - b(\theta_{-1})]}{(\theta_0 - \theta_{-1})} n + \frac{\log[\beta_0/(1-\alpha_0)]}{(\theta_0 - \theta_{-1})}. \tag{4.2}$$

Otherwise, sampling is continued. n_0 will be the value of n such that the lower boundary of the test of H_0 vs. H_1

will equal the upper boundary of the test of H_{-1} vs. H_0. This value can then be expressed in terms of the test parameters. n_0 will be the solution to the equation

$$\frac{[b(\theta_0) - b(\theta_{-1})]}{(\theta_0 - \theta_{-1})} n_0 + \frac{\log[(1-\beta_0)/\alpha_0]}{(\theta_0 - \theta_{-1})} =$$

$$\frac{[b(\theta_1) - b(\theta_0)]}{(\theta_1 - \theta_0)} n_0 + \frac{\log[\beta_1/(1 - \alpha_1]}{(\theta_1 - \theta_0)}. \qquad (4.3)$$

This implies

$$n_0 = \frac{(\theta_1 - \theta_0)\log[(1 - \beta_0)/\alpha_0] - (\theta_0 - \theta_{-1}) \log[\beta_1/(1 - \alpha_1)}{(\theta_0 - \theta_{-1})[b(\theta_1) - b(\theta_0)] - (\theta_1 - \theta_0)[b(\theta_0) - b(\theta_{-1})]}$$

(4.4)

The points a, c, and d on the X_n-axis (see Figure 12) are the values of the parallel lines at $n = n_0$ and can be determined by:

$$a = \frac{[b(\theta_1) - b(\theta_0)]}{(\theta_1 - \theta_0)} n_0 + \frac{\log[(1 - \beta_1)/\alpha_1]}{(\theta_1 - \theta_0)},$$

$$c = \frac{[b(\theta_1) - b(\theta_0)]}{(\theta_1 - \theta_0)} n_0 + \frac{\log[\beta_1/(1 - \alpha_1]}{(\theta_1 - \theta_0)},$$

$$c = \frac{[b(\theta_0) - b(\theta_{-1})]}{(\theta_0 - \theta_{-1})} n_0 + \frac{\log[(1 - \beta_0)/\alpha_0]}{(\theta_0 - \theta_{-1})},$$

and

$$d = \frac{[b(\theta_0) - b(\theta_{-1})]}{(\theta_0 - \theta_{-1})} n_0 + \frac{\log [(\beta_0/(1 - \alpha_0]}{(\theta_0 - \theta_{-1})}. \qquad (4.5)$$

Point c is given be two expressions because the two test boundaries intersect at this value. By having two equations equaling c, it becomes easy to solve equations (4.5) for the values α_0, α_1, β_0, and β_1 to obtain

$$\alpha_0 = \frac{1 - t_0(d)}{t_0(c) - t_0(d)},$$

$$\beta_0 = \frac{t_0(c)\ t_0(d) - t_0(d)}{t_0(c) - t_0(d)},$$

$$\alpha_1 = \frac{1 - t_1(c)}{t_1(a) - t_1(c)},$$

and $$\beta_1 = \frac{t_1(a)\ t_1(c) - t_1(c)}{t_1(a) - t_1(c)}, \qquad (4.6)$$

where

$$t_0(x) = \exp\{x(\theta_0 - \theta_{-1}) - [b(\theta_0) - b(\theta_{-1})]n_0\}$$

and $$t_1(x) = \exp\{x(\theta_1 - \theta_0) - [b(\theta_1) - b(\theta_0)]n_0\}.$$

The Operating Characteristic Function

Consider the operating characteristic function, $L(\theta)$, the probability of accepting H_0 given the value of θ. At $n = n_0$, H_0 will be accepted if

1. $c < X_n < a$ with CM being the first boundary to be crossed, or if
2. $d < X_n < c$ with CP being the first boundary to be crossed.

Let $L_1(\theta)$ and $L_{-1}(\theta)$ be the respective probabilities of 1 and 2 above. Then $L(\theta) = L_1(\theta) + L_{-1}(\theta)$. Lemmas 1 and 2 will derive $L_1(\theta)$ and $L_{-1}(\theta)$, respectively. Then Theorem 4.1 will give the general form of $L(\theta)$.

<u>Lemma 1</u>.

$$\begin{aligned} L_1(\theta) = {} & (\exp\{-h_0 a\} - \exp\{-h_0 c\})^{-1}\{\exp\{-h_0 a\} \\ & \times[G(\theta, n_0, a) - G(\theta, n_0, c)] \\ & - \exp\{n_0[b(\theta - h_0) - b(\theta)]\} [G(\theta - h_0, n_0, a) \\ & - G(\theta - h_0, n_0, c)]\} \quad \text{for } h_0 \neq 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} L_1(\theta) = {} & (a - c)^{-1} \{a [G(\theta, n_0, a) - G(\theta, n_0, c)] \\ & \int_c^a x g_n(x, \theta)\, dx\} \quad \text{for } h_0 = 0 \end{aligned} \tag{4.8}$$

where h_0 is the solution to $b(\theta - h) = b(\theta) - h (b(\theta_1) - b(\theta_0))/(\theta_1 - \theta_0)$.

<u>Proof</u>: Recall from Chapter II, equation (2.4), that the density function of X_n is

$$g_n(x_n, \theta) = \exp\{k_n(x_n) + \theta x_n - nb(\theta)\}.$$

Therefore

$$L_1(\theta) = \int_c^a g_n(x, \theta)\ \Pi(x)dx \quad \text{for } n = n_0 \tag{4.9}$$

where $\Pi(x)$ is the probability that, when starting at (n_0, x), $c < x < a$, the process crosses CM first.

Cox and Miller (1965), found for a random walk which operates between parallel boundaries $X_n = a*$ and $X_n = c*$ and starts at $X_0 = 0$, the approximate probability p_{-c*} that the walk ceases with absorption by $X_n = c*$ is

$$P_{-c*} = \frac{\exp\{-\lambda_0 a*\} - 1}{\exp\{-\lambda_0 a*\} - \exp\{\lambda_0 c*\}} \quad \text{for} \quad \lambda_0 \neq 0$$

$$= \frac{a*}{a* + c*} \quad \text{for} \quad \lambda_0 = 0, \tag{4.10}$$

where λ_0 is the nonzero solution for $E(\exp\{-\lambda x\}) = 1$.

P_{-c*} is a function not only of λ_0 and $c*$, but also of $a*$. Billard and Vagholkar transformed this three-dimensional function to $\Pi(x)$. $\Pi(x)$ is actually a three-dimensional function of a - x and b - x, and is found to be

$$\Pi(x) = \frac{\exp\{-h_0 a\} - \exp\{-h_0 x\}}{\exp\{-h_0 a\} - \exp\{-h_0 c\}} \quad \text{for} \quad h_0 \neq 0$$

$$= \frac{a - x}{a - c} \quad \text{for} \quad h_0 = 0 \qquad (4.11)$$

where h_0 is the solution for $E(\exp\{-h(X - \tan\psi)\}) = 1$.
For Koopman-Darmois densities,

$$E(\exp\{-hX\}) = \int_\Omega \exp\{k(x) + (\theta - h)x - b(\theta)\}\, dx$$

$$= \exp\{b(\theta - h) - b(\theta)\}. \qquad (4.12)$$

So $E(\exp\{-h(X - \tan\psi)\}) = 1$ can be restated as

$$\exp\{h \tan\psi - b(\theta) + b(\theta - h)\} = 1,$$

which implies that

$$b(\theta - h_0) = b(\theta) - h_0 \tan\psi. \qquad (4.13)$$

Remembering that $\tan\psi$ is the same as the slope of the line if ψ is the angle between the line and the n-axis,

$$\tan\psi = \frac{b(\theta_1) - b(\theta_0)}{\theta_1 - \theta_0}. \qquad (4.14)$$

Thus h_0 will be the nonzero solution to

$$b(\theta - h_0) = b(\theta) - h_0 \frac{b(\theta_1) - b(\theta_0)}{\theta_1 - \theta_0} . \qquad (4.15)$$

Now what remains is the calculation of

$$\int_c^a g_n(x, \theta)\Pi(x)dx, \quad \text{where} \quad n = n_0.$$

For convenience, define $G(\theta, n_0, k)$ at $n = n_0$ as

$$G(\theta, n_0, k) = \int_{-\infty}^{k} g_n(x, \theta)dx. \qquad (4.16)$$

Thus G is the cumulative distribution function for the density of X_n.

An expression for $L_1(\theta)$ can now be found using (4.11), for $h_0 \neq 0$; that is,

$$L_1(\theta) = \int_c^a g_n(x, \theta)\ \Pi(x)dx$$

$$= \int_c^a \exp\{k_n(x) + \theta x - n_0 b(\theta)\}[(\exp\{-h_0 a\}$$

$$- \exp\{-h_0 x\})/(\exp\{-h_0 a\} - \exp\{-h_0 c\})]\ dx.$$

This quantity can be expressed as

$$L_1(\theta) = (\exp\{-h_0a\} - \exp\{-h_0c\})^{-1}\{\exp\{-h_0a\}[G(\theta, n_0, a) - G(\theta, n_0, c)] - \exp\{n_0[b(\theta - h_0) - b(\theta)]\} \times [G(\theta - h_0, n_0, a) - G(\theta - h_0, n_0, c)]\}. \tag{4.17}$$

If $h_0 = 0$, then

$$L_1(\theta) = \int_c^a \exp\{k_n(x) + \theta x - n_0\, b(\theta)\}[(a-x)/(a-c)]\, dx$$

$$= (a - c)^{-1}\{a[G(\theta, n_0, a) - G(\theta, n_0, c)] - \int_c^a xq_n(x, \theta)dx\}. \tag{4.18}$$

The proof is now complete.

<u>Lemma 2</u>.

$$L_{-1}(\theta) = (\exp\{-h_0'c\} - \exp\{-h_0'd\})^{-1} \times \{\exp(n_0(b(\theta - h_0') - b(\theta))\}[G(\theta - h_0', n_0, c) - G(\theta - h_0', n_0, c)] - \exp\{-h_0'd\}[G(\theta, n_0, c) - G(\theta, n_0, d)]\}, \quad h_0' \neq 0 \tag{4.19}$$

$$L_{-1}(\theta) = (c - d)^{-1}\{\int_d^c xq_n(x, \theta)dx - c[G(\theta, n_0, c) - G(\theta, n_0, d]\} \tag{4.20}$$

where h_0' is the solution of

$$b'(\theta - h') = b(\theta) - h'[b(\theta_0) - b(\theta_{-1})] / (\theta_0 - \theta_{-1})$$

Proof: In order to find $L_{-1}(\theta)$, it is necessary to find $\Gamma(x)$, the approximate probability that the process crosses CP first when starting at (n_0, x), $d < x < c$. Using (4.9),

$$\Gamma(x) = \frac{\exp\{-h_0'x\} - \exp\{-h_0'd\}}{\exp\{-h_0'c\} - \exp\{-h_0'd\}} \quad \text{for} \quad h_0' \neq 0$$

$$= \frac{x - d}{c - d} \quad \text{for} \quad h_0' = 0 \qquad (4.21)$$

Therefore, for h_0'

$$L_{-1}(\theta) = \int_d^c g_n(x, \theta)\ \Gamma(x)\ dx$$

$$= \int_d^c \exp\{k_n(x) + \theta x - n_0 b(\theta)\}[(\exp\{-h_0'x\} - \exp\{-h_0'd\})/(\exp\{-h_0'c\} - \exp\{-h_0'd\})]\ dx$$

$$= (\exp\{-h_0'c\} - \exp\{-h_0'd\})^{-1}\ \{\exp\{n_0(b(\theta - h_0') - b(\theta))\}[G(\theta - h_0', n_0, c) - G(\theta - h_0', n_0, d)] - \exp\{-h_0'd\}[G(\theta, n_0, c) - G(\theta, n_0, d)]\}. \qquad (4.22)$$

For $h_0' = 0$,

$$L_{-1}(\theta) = \int_d^c \exp\{k_n(x) + \theta x - n_0 b(\theta)\}[(x-c)/(c-d)]\ dx$$

implies

implies

$$L_{-1}(\theta) = (c - d)^{-1}\{\int_d^c xg_n(x, \theta)dx - c[G(\theta, n_0, c) - G(\theta, n_0, d)]\}. \qquad (4.23)$$

The following theorem is presented with the previous work serving as proof:

<u>Theorem</u> <u>4.1</u>: Let x_i, $i = 1, 2, \ldots$ be random observations from $f_\theta(x) = \exp\{k(x) + \theta x - b(\theta)\}$, and let Figure 12 serve as the sampling region for testing $H_{-1} : \theta = \theta_{-1}$ vs. $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$. Let $L(\theta)$ be the approximate probability of accepting H_0 given θ (i.e., OC function), then

$$L(\theta) = L_1(\theta) + L_{-1}(\theta),$$

where $L_1(\theta)$ and $L_{-1}(\theta)$ are defined in Lemmas 1 and 2, respectively.

Average Sample Number Function

Consider the ASN function, which is the expected number of observations required to reach a decision as a function of θ. Three possibilities exist for this procedure:

1. A decision is reached at $n = n_0$, or

2. $c < X_n < a$ at $n = n_0$, $N_1(\theta)$ denoting the average sample number beyond $n_0 = n$ for deciding in favor of either H_1 or H_0, or

3. $d < X_n < c$ at $n = n_0$, $N_{-1}(\theta)$ denoting the average sample number beyond $n = n_0$ for deciding in favor of either H_0 or H_{-1}.

The following theorem provides the ASN function for the proposed testing procedure.

<u>Theorem</u> <u>4.2</u>: Let the assumptions of Theorem 4.1 hold. The Average Sample Number function, $E_\theta(N)$, is

$$E_\theta(N) = n_0 + N_1(\theta) + N_{-1}(\theta),$$

where

$$\begin{aligned} N_1(\theta) = {} & (b'(\theta) - \tan\psi)^{-1} \{(a - c)\exp\{n_0[b(\theta - h_0) \\ & - b(\theta)]\}\,[G(\theta - h_0, n_0, a) - G(\theta - h_0, n_0, c)] \\ & \div (\exp\{-h_0 a\} - \exp\{-h_0 c\}) + (c\cdot\exp\{-h_0 a\} \\ & - a\cdot\exp\{-h_0 c\})[G(\theta, n_0, a) - G(\theta, n_0, c)] \\ & \div (\exp\{-h_0 a\} - \exp\{-h_0 c\}) \\ & - \int_c^a x g_n(x, \theta)\,dx\} \quad \text{for} \quad h_0 \neq 0, \end{aligned}$$

$$\begin{aligned} N_1(\theta) = {} & [b''(\theta)]^{-1}\{(a + c)\int_c^a x g_n(x, \theta)\,dx - ac \\ & \times [G(\theta, n_0, a) - G(\theta, n_0, c)] \end{aligned}$$

$$N_1(\theta) = [b''(\theta)]^{-1}\{(a + c) \int_c^a xg_n(x, \theta)dx - ac$$
$$\times [G(\theta, n_0, a) - G(\theta, n_0, c)]$$
$$- \int_c^a x^2 g_n(x, \theta)dx\} \quad \text{for} \quad h_0 = 0,$$

and

$$N_{-1}(\theta) = (b'(\theta) - \tan\phi)^{-1}\{(c - d)\exp\{n_0[b(\theta - h_0')$$
$$- b(\theta)]\}[G(\theta - h_0', n_0, c)$$
$$- G(\theta - h_0', n_0, d)]/(\exp\{-h_0'c\} - \exp\{-h_0'd\})$$
$$+ (d\cdot\exp\{-h_0'c\} - c\cdot\exp\{-h_0'd\})[G(\theta, n_0, c)$$
$$- G(\theta, n_0, d)]/(\exp\{-h_0'c\} - \exp\{-h_0'd\})$$
$$- \int_d^c xg_n(x, \theta)dx\} \quad \text{for} \quad h_0' \neq 0,$$

$$N_{-1}(\theta) = [b''(\theta)]^{-1}\{(c + d) \int_d^c xg_n(x, \theta)dx$$
$$- cd[G(\theta, n_0, c) - G(\theta, n_0, d)]$$
$$- \int_d^c x^2 g_n(x, \theta)dx\} \quad \text{for} \quad h_0'=0$$

<u>Proof</u>:

Since no decision will be made before n_0, the average sample number will be n_0 plus the average sample number beyond $n = n_0$ for deciding in favor of either H_1 or H_0 plus the average sample number beyond $n = n_0$ for deciding in favor of either H_0 or H_{-1}. The ASN function, denoted by $E_\theta(N)$, will be

$$E_\theta(N) = n_0 + N_1(\theta) + N_{-1}(\theta). \tag{4.24}$$

Let $N_1(\theta)$ be

$$N_1(\theta) = \int_c^a g_n(x, \theta)\ n(x)dx, \qquad (4.25)$$

where n(x) is the expected number of observations to absorption for a random walk starting at point $X_n = x$ at $n = n_0$ and operating between the parallel lines AL and CM.

Cox and Miller found that the expected number of steps for a random walk starting at zero and operating between parallel absorbing boundaries $X_n = a*$ and $X_n = c*$ is

$$E(N) = \frac{(a* + c*) - a* \exp\{\lambda_0 c*\} - c* \exp\{-\lambda_0 a*\}}{\mu(\exp\{-\lambda_0 a*\} - \exp\{\lambda_0 c*\})} \ ; \ \lambda_0 \neq 0$$

$$= \frac{a*\ c*}{\sigma^2} \ ; \quad \lambda_0 = 0, \qquad (4.26)$$

where λ_0 is the nonzero solution of $E(\exp\{-\lambda X\}) = 1$. Billard and Vagholkar transformed (4.22) into

$$n(x) = \left[\frac{(a - c)\exp\{-h_0 x\} - a\cdot\exp\{-h_0 c\} + c\cdot\exp\{-h_0 x\}}{\exp\{-h_0 a\} - \exp\{-h_0 c\}} - x\right]$$

$$\div\ (\mu - \tan\psi) \quad \text{for } h_0 \neq 0,$$

$$= (a - x)(x - c)/\sigma^2 \quad \text{for } h_0 = 0 \qquad (4.27)$$

where h_0 is the nonzero solution of

$$E(\exp\{-h(x - \tan\psi)\}] = 1.$$

For the Koopman-Darmois family of densities

$$n(x) = \left[\frac{(a - c)\exp\{-h_0 x\} - a\cdot\exp\{-h_0 c\} + c\cdot\exp\{-h_0 a\}}{\exp\{-h_0 a\} - \exp\{-h_0 c\}} - x\right]$$

$$\div\ (b'(\theta) - \tan\psi) \quad \text{for } h_0 \neq 0,$$

$$= (a - x)(x - c)/b''(\theta) \quad \text{for } h_0 = 0, \tag{4.28}$$

where h_0 is such that

$$b(\theta - h_0) = b(\theta) - h_0 \frac{b(\theta_1) - b(\theta_0)}{\theta_1 - \theta_0}.$$

$N_1(\theta)$ will then be

$$N_1(\theta) = \int_c^a \exp\{k_n(x) + \theta x - n_0 b(\theta)\}\ n(x)\ dx$$

$$\begin{aligned} &= (b'(\theta) - \tan\psi)^{-1}[(\exp\{-h_0 a\} - \exp\{-h_0 c\})^{-1} \\ &\quad \times \int_c^a [(a - c)\exp\{-h_0 x\} - a\cdot\exp\{-h_0 c\} \\ &\quad + c\cdot\exp\{-h_0 a\}]g_n(x, \theta)\ dx \\ &\quad - \int_c^a x g_n(x, \theta)\ dx, \end{aligned}$$

$$= (b'(\theta) - \tan\psi)^{-1}\{(a - c)\exp\{n_0[b(\theta - h_0) - b(\theta)]\}[G(\theta - h_0, n_0, a) - G(\theta - h_0, n_0, c)] \div (\exp\{-h_0 a\} - \exp\{-h_0 c\}) + (c\cdot\exp\{-h_0 a\} - a\cdot\exp\{-h_0 c\})[G(\theta, n_0, a) - G(\theta, n_0, c)] \div (\exp\{-h_0 a\} - \exp\{-h_0 c\}) - \int_c^a x g_n(x, \theta)dx\}; \quad h_0 \neq 0. \tag{4.29}$$

For $h_0 = 0$,

$$N_1(\theta) = \int_c^a \exp\{k_n(x) + \theta x - n_0 b(\theta)\}\frac{(a - x)(x - c)}{b''(\theta)}\, dx$$

$$= (b''(\theta))^{-1}\{(a + c)\int_c^a x g_n(x, \theta)\, dx - ac[G(\theta, n_0, a) - G(\theta, n_0, c)] - \int_c^a x^2 g_n(x, \theta)\, dx\}. \tag{4.30}$$

Due to symmetry, $N_{-1}(\theta)$ can be found by replacing in $N_1(\theta)$ a with c, c with d, ψ with ϕ, and h_0 with h_0'. Therefore,

$$N_{-1}(\theta) = (b'(\theta) - \tan\phi)^{-1}\{(c - d)\exp\{n_0[b(\theta - h_0') - b(\theta)]\}[G(\theta - h_0', n_0, c) - G(\theta - h_0', n_0, d)]/(\exp\{-h_0' c\} - \exp\{-h_0' d\}) + (d\cdot\exp\{-h_0' c\} - c\cdot\exp\{-h_0' d\})[G(\theta, n_0, c) - G(\theta, n_0, d)]/(\exp\{-h_0' c\} - \exp\{-h_0' d\}) - \int_d^c x g_n(x, \theta)\, dx\} \quad \text{for } h_0' \neq 0, \tag{4.31}$$

and

$$N_{-1}(\theta) = [b''(\theta)]^{-1}\{(c + d) \int_d^c x g_n(x, \theta)\, dx$$
$$- cd[G(\theta, n_0, c) - G(\theta, n_0, d)]$$
$$- \int_d^c x^2 g_n(x, \theta)\, dx\} \quad \text{for } h_0' = 0. \tag{4.32}$$

The proof of Theorem 4.2 is now complete.

Approximations for the OC and ASN functions have been developed for the procedure depicted in Figure 12. In the next section, the OC function will be used in part to adjust the error rates to obtain a more desirable test.

Error Rate Adjustment

Suppose it is desired to test

$$H_{-1} : \theta = \theta_{-1} \;,$$
$$H_0 : \theta = \theta_0 \;,$$
and
$$H_1 : \theta = \theta_1 \;, \tag{4.33}$$

where θ is a parameter from $f_\theta(x) = \exp\{k(x) + \theta x - b(\theta)\}$. As mentioned in Chapter III, when two SPRTs are combined to test these hypotheses, the error rates α_0, α_1, β_0, β_1 used in the SPRTs do not result in desired error levels. It is the goal of this section to find the values of the error rates that do give desired results.

The approach used to adjust the error rates will be to set values to the probabilities below:

$$P(\text{accepting } H_0|\theta = \theta_1) = \gamma_1,$$
$$P(\text{accepting } H_0|\theta = \theta_{-1}) = \gamma_2,$$
$$P(\text{accepting } H_1|\theta = \theta_0) = \gamma_3,$$
and
$$P(\text{accepting } H_{-1}|\theta = \theta_0) = \gamma_4. \tag{4.34}$$

Since the probability that H_{-1} is accepted when H_1 is true will usually be very small, γ_1 can be thought of as the probability of error when θ_1 is the true value of θ. Likewise γ_2 can be the probability of error when H_{-1} is true. The quantity $(\gamma_3 + \gamma_4)$ will be the probability of error when H_0 is true. It will now be necessary to find the probabilities associated with equation (4.34).

<u>Theorem</u> <u>4.3</u>: Based on (4.34), the four desired error rates are

$$L_1(\theta_1) + L_{-1}(\theta_1) = \gamma_1,$$
$$L_1(\theta_{-1}) + L_{-1}(\theta_{-1}) = \gamma_2,$$
$$1 - G(\theta_0, n_0, c) - L_1(\theta_0) = \gamma_3,$$
and
$$G(\theta_0, n_0, c) - L_{-1}(\theta_0) = \gamma_4. \tag{4.35}$$

<u>Proof</u>: Theorem 4.1 found an approximation for the Operating Characteristic function, which is the probability of accepting H_0 given a value of θ. Thus

$$P(\text{accepting } H_0|\theta) = L(\theta) = L_1(\theta) + L_{-1}(\theta). \tag{4.36}$$

Accepting H_1 will require, at $n = n_0$, $X_n > c$ and the process not crossing boundary CM first (see Figure 12). The random variable X_n will be greater than c with probability

$$P(X_n > c) = \int_c^{\infty} g_n(x, \theta)\ dx, \tag{4.37}$$

and CM will be crossed first with probability

$$P(\text{CM crossed first}) = L_1(\theta). \tag{4.38}$$

Combining (4.37) and (4.38) will give

$$\begin{aligned} P(\text{accepting } H_1|\theta) &= \int_c^{\infty} g_n(x, \theta)\ dx - L_1(\theta) \\ &= 1 - G(\theta, n_0, c) - L_1(\theta). \end{aligned} \tag{4.39}$$

Likewise, the probability of accepting H_{-1} can be expressed as

$$\begin{aligned} P(\text{accepting } H_{-1}|\theta) &= \int_{-\infty}^{c} g_n(x, \theta)\ dx - L_{-1}(\theta) \\ &= G(\theta, n_0, c) - L_{-1}(\theta). \end{aligned} \tag{4.40}$$

Using (4.34), the four desired error rates become

$$L_1(\theta_1) + L_{-1}(\theta_1) = \gamma_1,$$

$$L_1(\theta_{-1}) + L_{-1}(\theta_{-1}) = \gamma_2,$$

$$1 - G(\theta_0, n_0, c) - L_1(\theta_0) = \gamma_3,$$

and

$$G(\theta_0, n_0, c) - L_{-1}(\theta_0) = \gamma_4. \tag{4.41}$$

The proof is complete.

The four equations (4.41) are all equations in a, c, d, and n_0, which are functions of α_0, α_1, β_0, β_1 (see equation 4.6). It is desired to set these error rates such that the desired error rates are attained. Therefore, the problem is to find the a, c, d, and n_0 that provide a solution to the system of equations denoted by (4.41). A method for solving a system of nonlinear equations using SAS is presented in Chapter V.

This process (Figure 12) is similar to Billard and Vagholkar's (1969), and many of their techniques were utilized in this dissertation. There are some notable differences between the two methods, however. The procedure defined in this chapter adjusts the nominal error rates of the two SPRTs to obtain the desired error probabilities. Billard and Vagholkar's geometric approach begins with the desired error probabilities and then determines the geometric parameters that minimize the ASN function at a given point.

A minimum sample size, n_0, is considered for this procedure. This is useful for researchers not willing to stop sampling after only a few observations. Placing a minimum on sample size is not a necessity or requirement of this method, however. The parallel boundaries that characterize the sampling regions for $n > n_0$ can be extended to meet the X_n-axis (Figure 13). This extension will have little effect on the error rates, since making a decision early in the process will happen infrequently. Since no decision can be made in Figure 12 before it is made in Figure 13, the ASN functions for the method in Figure 13 will be smaller than that depicted in Figure 12. Thus, the extension of the parallel boundaries would seem prudent, and Chapter V compares the two methods.

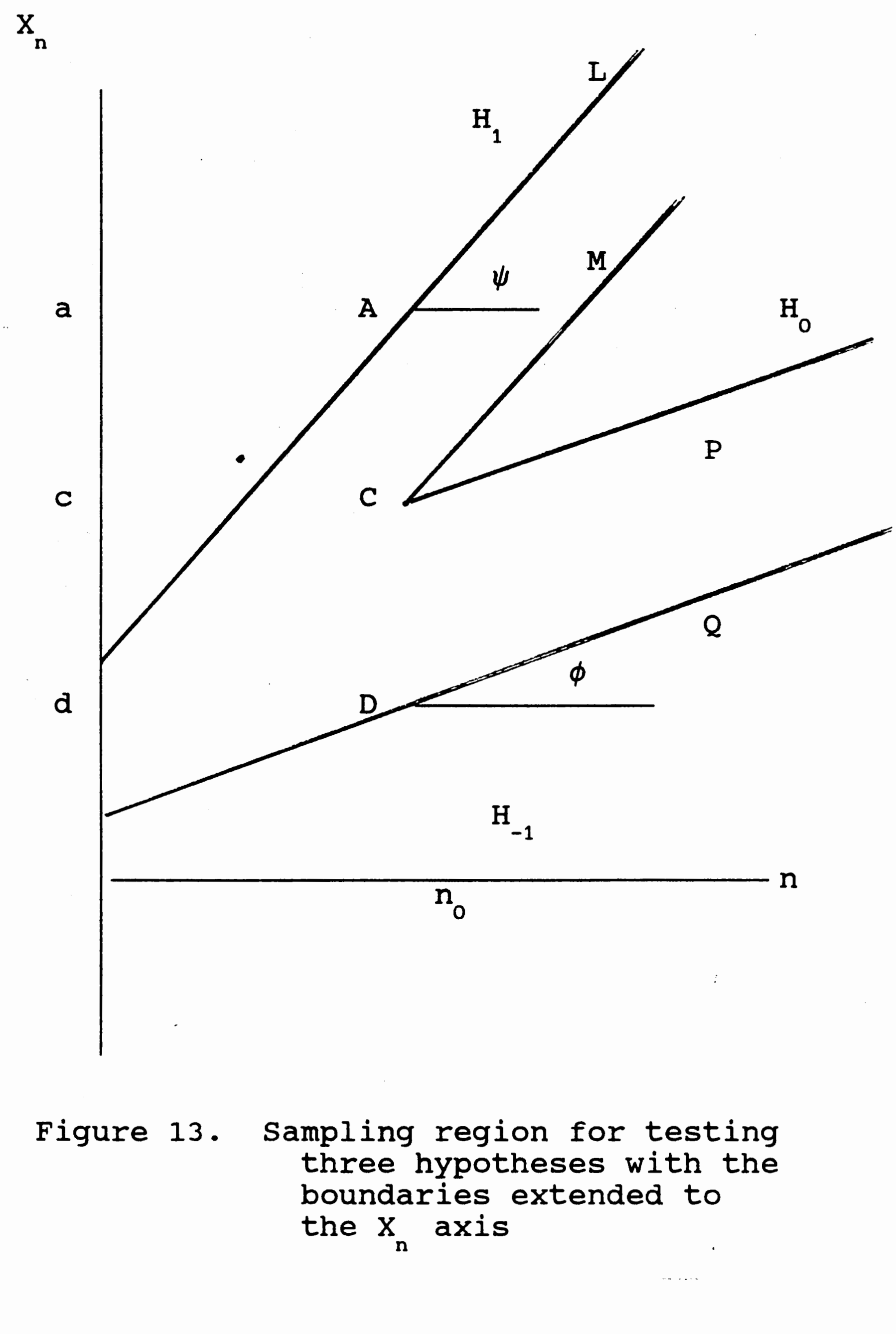

Figure 13. Sampling region for testing three hypotheses with the boundaries extended to the X_n axis

CHAPTER V

TWO EXAMPLES WITH MONTE CARLO RESULTS

In order to examine the effectiveness of the procedure developed in Chapter IV, it will be derived for certain tests and compared using simulation to other known methods. The two distributions that will be used are the exponential and normal, which are commonly used in the literature for sequential sampling (e.g., Huffman, 1983, Billard and Vagholkar, 1969). One should note, however, that the procedure presented in this dissertation is applicable to any distribution in the Koopman-Darmois family.

A Method Using SAS to Solve a System Of Nonlinear Equations

The method for sequentially testing three hypotheses mentioned in Chapter IV relied primarily on solving a system of four non-linear equations in four unknowns. There exist many computing techniques to solve such a system. However, the equations included in this dissertation involve cumulative distribution functions. This creates problems for some routines that must estimate the incomplete integrals that accompany such functions. SAS is convenient for solving these systems because it has implicit functions for the cumulative distribution function of most known distributions.

The ETS version of SAS has a procedure named PROC SYSNLIN for solving a system of nonlinear equations. An alternative method for SAS without ETS has been developed using PROC NLIN, SAS's nonlinear regression procedure. Consider the following general example of four equations, four unknowns.

Let x_1, x_2, x_3, and x_4, be four unknowns, f_1, f_2, f_3, and f_4 be four functions of x_1 through x_4, and let c_1, c_2, c_3, and c_4 be four constants. The system may take the appearance of:

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) &= c_1 \\ f_2(x_1, x_2, x_3, x_4) &= c_2 \\ f_3(x_1, x_2, x_3, x_4) &= c_3 \\ f_4(x_1, x_2, x_3, x_4) &= c_4 \end{aligned} \tag{5.1}$$

The values of x_1, x_2, x_3, and x_4 that solve the previous system can be found using the following SAS code.

```
INPUT A  B  C  D  Y  ;
CARDS;
1  0  0  0  c1
0  1  0  0  c2
0  0  1  0  c3
0  0  0  1  c4
;
PROC NLIN;
MODEL Y = f1*A + f2*B + f3*C + f4*D;
RUN;
```

The quantities c_1 through c_4 will be numeric values and the model statement can be quite complicated depending on the system. Between the PROC NLIN and MODEL statements should be a PARMS statement, giving initial parameter estimates, a BOUNDS statement, placing restrictions on the unknowns.

This method is used to solve systems of equations in this chapter.

Test for Exponential Parameter

Let $x_1, x_2, \ldots, x_n, \ldots$ be random observations from an exponential distribution with parameter λ; that is,

$$f(x) = \lambda e^{-\lambda x} = \exp\{\log \lambda - \lambda x\}. \tag{5.2}$$

Therefore, using the definition of Koopman-Darmois densities, $\theta = -\lambda$, $b(\theta) = -\log(-\theta)$, and $b'(\theta) = -1/\theta$. Also note that the mean of the distribution, μ, is equal to $1/\lambda$.

Suppose it is desired to test

$$H_{-1}: \mu = 1,$$
$$H_0 : \mu = 2,$$
and
$$H_1 : \mu = 3, \tag{5.3}$$

or equivalently,

$$H_{-1}: \theta = -1,$$
$$H_0 : \theta = -1/2,$$
and
$$H_1 : \theta = -1/3. \tag{5.4}$$

Assume the desired error probability is set at 0.10. The distribution of X_n will be gamma(n, λ); that is,

$$g_\theta(x) = \exp\{-\log(\Gamma(n)) + (n - 1)\cdot\log x + \theta x + n\cdot\log(-\theta)]. \tag{5.5}$$

Therefore,

$$k_n(x) = (n - 1) \log(x) - \log(\Gamma(n)).$$

h_0 is the solution to

$$\log(h - \theta) - \log(-\theta) - 6h\cdot(\log 3 - \log 2) = 0,$$

and h'_0 is the solution to

$$\log(h' - \theta) - \log(-\theta) - 2h'(\log 2) = 0$$

The values of h_0 and h'_0 must be estimated iteratively because the equations cannot be solved explicitly for these terms. For $\theta = \theta_{-1}$, h_0 to four decimal places of accuracy is -0.8834. h'_0 for $\theta = -1$ is $-1/2$. For $\theta = \theta_0 = -1/2$, $h'_0 = -1/6$, $h'_0 = 1/2$. For $\theta = \theta_1 = -1/3$, $h_0 = 1/6$, $h'_0 = 1$. The slopes of the upper and lower sets of parallel lines are

$$\text{slope}_1 = \frac{b(\theta_1) - b(\theta_0)}{\theta_1 - \theta_0} = 2.4328$$

and
$$\text{slope}_0 = \frac{b(\theta_0) - b(\theta_{-1})}{\theta_0 - \theta_{-1}} = 1.3863, \tag{5.6}$$

respectively.

The problem remains to find values of a, c, d, and n_0 that solve the system of equations associated with this test. Using Armitage's (1947) method for testing these hypotheses, the following values result:

$$a = 59.21,$$
$$c = 28.36,$$
$$d = 18.08,$$
and
$$n_0 = 17.21. \tag{5.7}$$

These values will be used as initial values in the computer iterations. The specified error rates used in the procedure above are

$$
\begin{aligned}
&\alpha_0 = 0.10,\\
&\beta_0 = 0.05,\\
&\alpha_1 = 0.05,\\
\text{and}\quad &\beta_1 = 0.10.
\end{aligned}
\tag{5.8}
$$

The system of equations associated with (5.4) is

$$
\begin{aligned}
&L_1(\theta_1) + L_{-1}(\theta_1) = 0.10\\
&L_1(\theta_{-1}) + L_{-1}(\theta_{-1}) = 0.10\\
&1 - G(\theta_0, n_0, c) - L_1(\theta_0) = 0.05\\
&G(\theta_0, n_0, c) - L_{-1}(\theta_0) = 0.05
\end{aligned}
\tag{5.9}
$$

where

$$
G(\theta, n, c) = \int_0^c \frac{(-\theta)^n}{\Gamma(n)} x^{n-1} \exp\{\theta x\}dx. \tag{5.10}
$$

For $\theta = \theta_{-1} = -1$, one can find

$$
\begin{aligned}
&-h_0 = 0.88342\\
&b(\theta_{-1} - h_0) = 2.149175\\
&\theta_{-1} - h_0 = -0.11658.
\end{aligned}
\tag{5.11}
$$

For convenience, define the above quantities as r_1, r_2, and

r_3, respectively. Using appropriate definitions for L_1 and L_{-1}, the system of equations denoted by (5.9) becomes

$$(\exp\{-a/6\} - \exp\{-c/6\})^{-1}\{\exp\{-a/6\}[G(-1/3, n_0, a) - G(-1/3, n_0, c)] - \exp\{n_0(\log 2 - \log 3)\} \times [G(-1/2, n_0, a) - G(-1/2, n_0, c)]\} + (\exp\{-c\} - \exp\{-d\})\{\exp\{-n_0\log 4\}[G(-4/3, n_0, c) - G(-4/3, n_0, d)] - \exp\{-c\}[G(-1/3, n_0, c) - G(-1/3, n_0, d)]\} = 0.10$$

$$(\exp\{r_1a\} - \exp\{r_1c\})^{-1}\{\exp\{r_1a\}[G(-1, n_0, a) - G(-1, n_0, c)] - \exp\{n_0r_2\}[G(r_3, n_0, a) - G(r_3, n_0, c]\} + (\exp\{c/2\} - \exp\{d/2\})^{-1} \times\{\exp\{n_0\log 2\}[G(-1/2, n_0, c) - G(-1/2, n_0, d)] - \exp\{c/2\}[G(-1, n_0, c) - G(-1, n_0, d)]\} = 0.10$$

$$1 - G(-1/2, n_0, c) - (\exp\{a/6\} - \exp\{c/6\})^{-1} \times \{\exp\{a/6\}[G(-1/2, n_0, a) - G(-1/2, n_0, c)] - \exp\{n_0(\log 3 - \log 2)\}[G(-1/3, n_0, a) - G(-1/3, n_0, c)]\} = 0.05$$

$$G(-1/2, n_0, c) - (\exp\{-c/2\} - \exp\{-d/2\})^{-1} \times \{\exp\{-n_0\log 2\}[G(-1, n_0, c) - G(-1, n_0, d)] - \exp\{-d/2\}[G(-1/2, n_0, c) - G(-1/2, n_0, d)]\} = 0.05. \quad (5.12)$$

Solving the system of equations, the values obtained are

$$a = 50.41,$$

$$c = 20.94,$$

$$d = 14.89,$$

and $$n_0 = 13.96. \tag{5.13}$$

These values result from adjusting the error rates associated with the two SPRTs that will be conducted simultaneously to obtain the desired probabilities of error.

Figure 14 represents this method using a minimum sample number, n_0, and Figure 15 is the same procedure without the minimum on the sample size. Figure 16 represents the procedure's comparison to Armitage's method. Note that the parallel boundaries are closer in the proposed method than Armitage's. This is a result of Armitage's procedure being too conservative. A Monte Carlo simulation of 2000 trials per parameter value was performed on PC SAS to compare these two procedure with Armitage's. Tables X, XI, and XII present the probabilities of accepting the hypotheses and average sample numbers given the mean of the exponential for all three procedures. Figures 17 and 18 are graphs of the empirical ASN functions.

The new procedures appear to be improvements over Armitage's. The error rates at the hypothesized values of θ are closer to the specified levels than the conservative error rates obtained by Armitage. This allows for decision making to occur sooner on the average, thus lending to smaller average sample numbers for all values of θ.

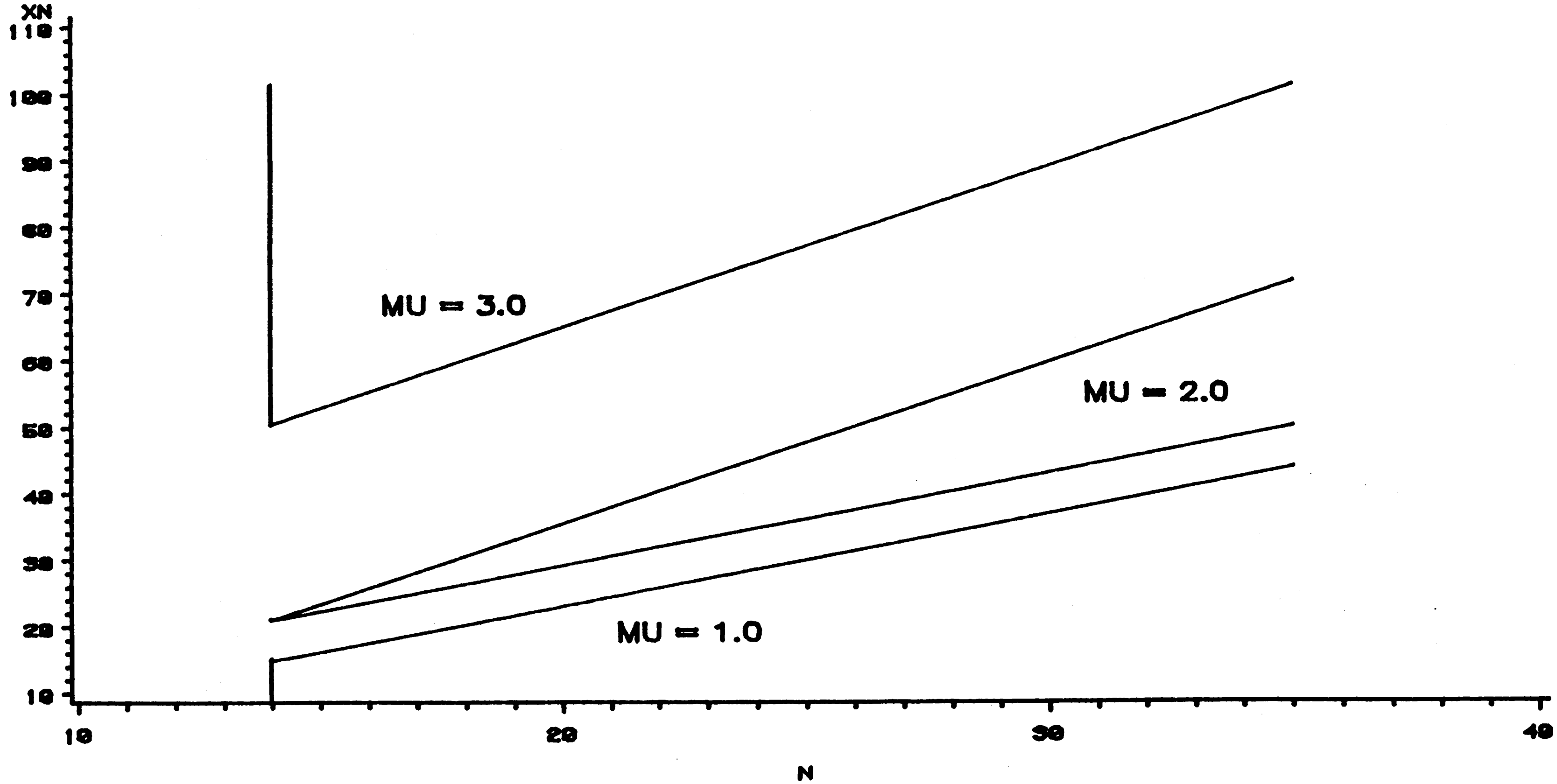

Figure 14. Sampling Region for Testing Three Values for the Mean of the Exponential Distribution (1.0, 2.0, 3.0)

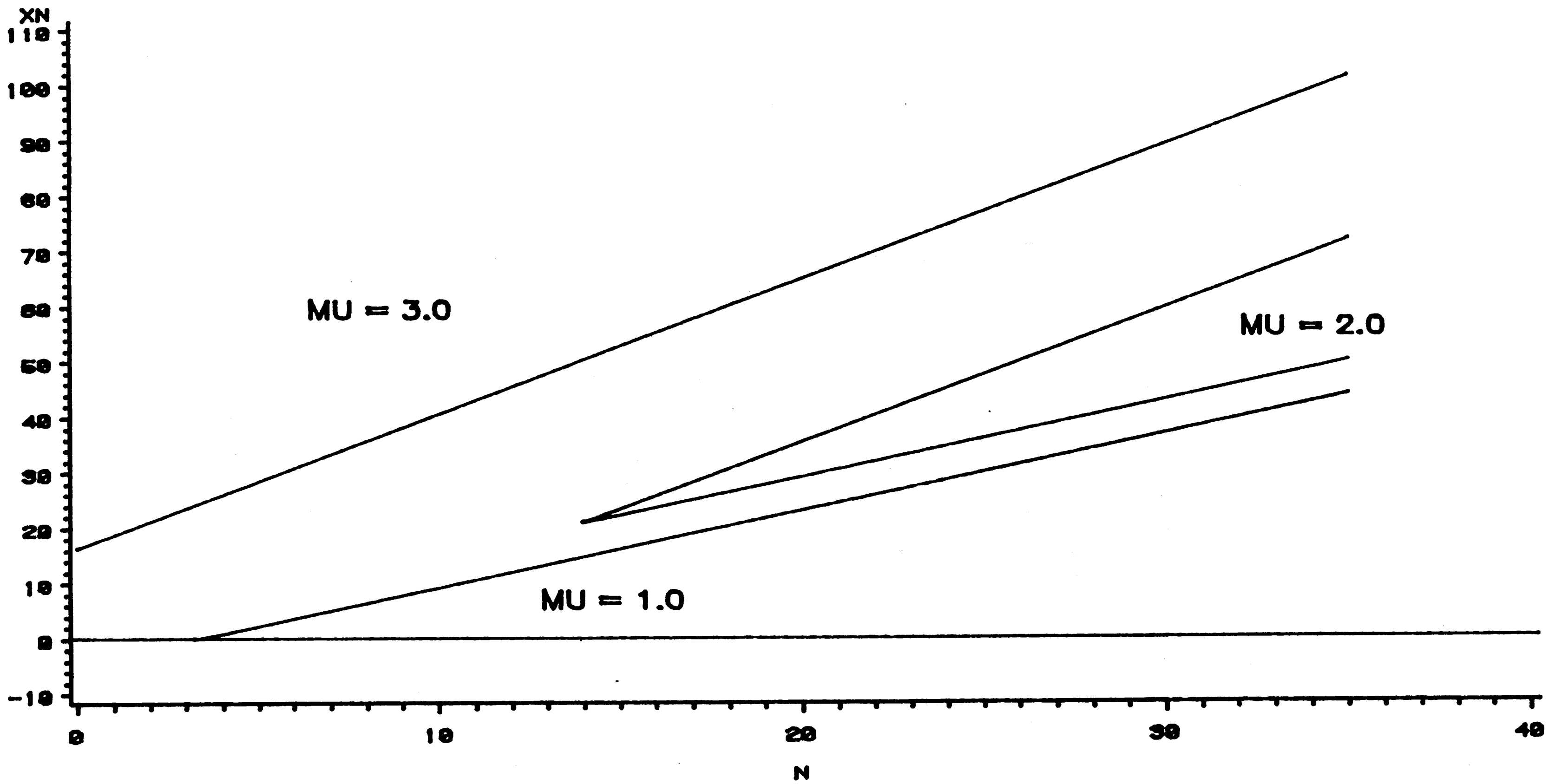

Figure 15. Sampling Region for Testing Three Values for the Mean of the Exponential Distribution (1.0, 2.0, 3.0), Boundaries Extended

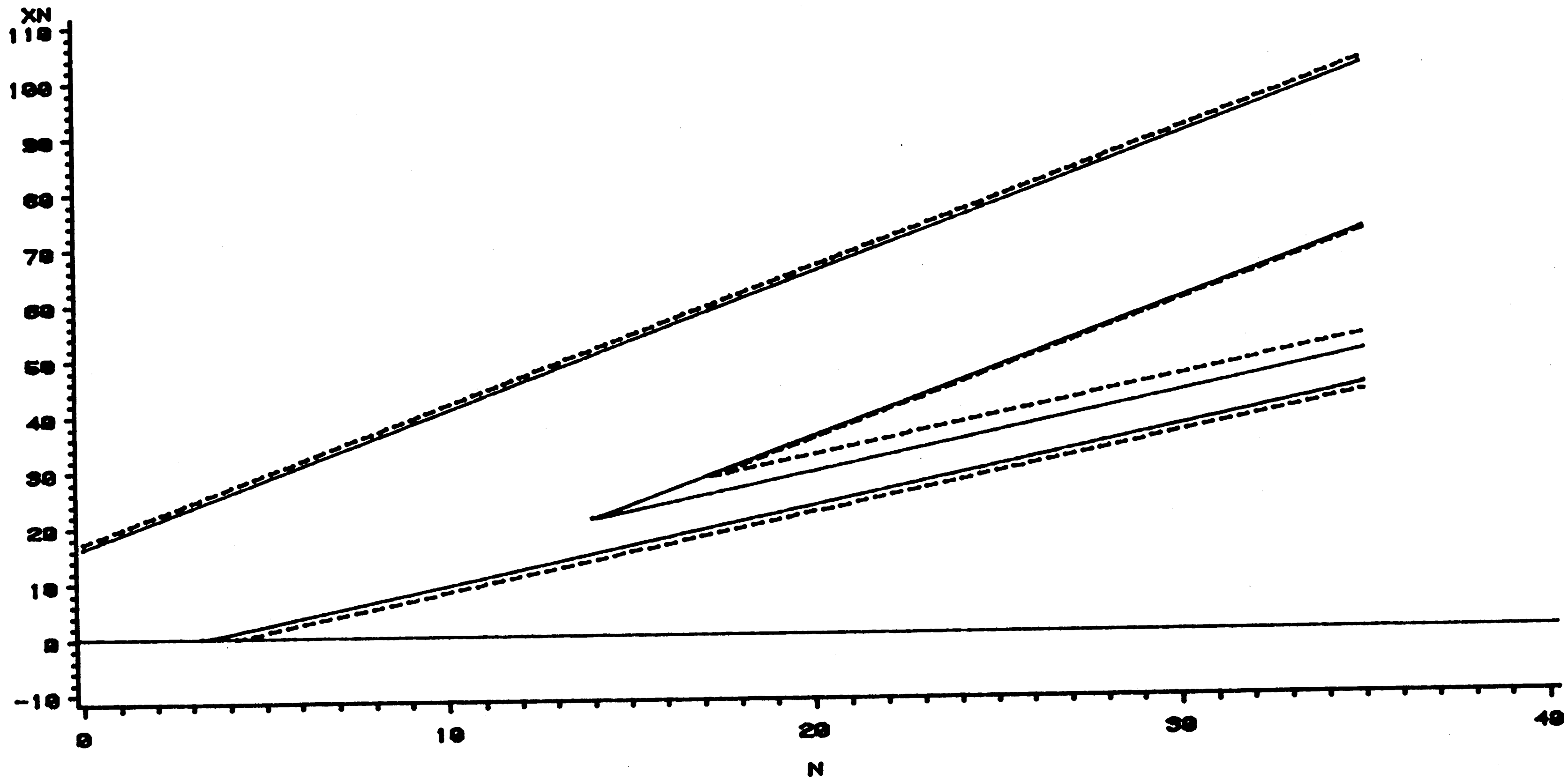

Figure 16. Armitage's (---) and Proposed (——) Sampling Regions for Testing Three Values for the Exponential Mean (1.0, 2.0, 3.0)

TABLE X

ERROR PROBABILITIES AND ASN VALUES FOR ARMITAGE'S TEST OF THREE VALUES FOR THE MEAN OF AN EXPONENTIAL DENSITY (1.0, 2.0, 3.0)

MEAN	Probability of Accepting H_{-1}	H_0	H_1	ASN
0.7	1.0000	0.0000	0.0000	9.3685
0.8	0.9995	0.0005	0.0000	10.6395
0.9	0.9970	0.0030	0.0000	12.9070
1.0	0.9820	0.0180	0.0000	15.7175
1.1	0.9285	0.0715	0.0000	19.4970
1.2	0.8390	0.1310	0.0000	22.1755
1.3	0.6725	0.3275	0.0000	24.3465
1.4	0.4850	0.5150	0.0000	25.5970
1.5	0.3560	0.6435	0.0005	26.0015
1.6	0.2275	0.7715	0.0010	25.7145
1.7	0.1450	0.8530	0.0020	26.0490
1.8	0.0900	0.9040	0.0060	27.5745
1.9	0.0585	0.9270	0.0145	30.2340
2.0	0.0435	0.9235	0.0330	33.4795
2.1	0.0285	0.8970	0.0745	37.2850
2.2	0.0190	0.8450	0.1360	42.2890
2.3	0.0130	0.7500	0.2370	46.3410
2.4	0.0105	0.5995	0.3900	49.3100
2.5	0.0075	0.4760	0.5165	49.1815
2.6	0.0025	0.3525	0.6450	44.9460
2.7	0.0045	0.2330	0.7625	41.5355
2.8	0.0015	0.1795	0.8190	38.7565
2.9	0.0030	0.1305	0.8665	34.1110
3.0	0.0025	0.0800	0.9175	30.0775
3.1	0.0025	0.0505	0.9470	27.1255
3.2	0.0005	0.0410	0.9585	25.0730
3.3	0.0005	0.0205	0.9790	22.6350

TABLE XI

ERROR PROBABILITIES AND ASN VALUES FOR PROPOSED TEST OF THREE VALUES FOR THE MEAN OF AN EXPONENTIAL DENSITY (1.0, 2.0, 3.0)

MEAN	Probability of Accepting H_{-1}	H_0	H_1	ASN
0.7	1.0000	0.0000	0.0000	14.1035
0.8	0.9980	0.0020	0.0000	14.3740
0.9	0.9770	0.0230	0.0000	15.0180
1.0	0.9415	0.0585	0.0000	16.1265
1.1	0.8345	0.1655	0.0000	17.1735
1.2	0.7205	0.2795	0.0000	17.8550
1.3	0.5800	0.4200	0.0000	18.6735
1.4	0.4265	0.5735	0.0000	19.3220
1.5	0.3070	0.6925	0.0000	19.8560
1.6	0.2185	0.7800	0.0015	20.5265
1.7	0.1420	0.8555	0.0025	22.5375
1.8	0.0645	0.9005	0.0045	24.3755
1.9	0.0465	0.9190	0.0165	27.8565
2.0	0.0305	0.9160	0.0375	31.2475
2.1	0.0305	0.8895	0.0800	35.3710
2.2	0.0205	0.8400	0.1395	40.5715
2.3	0.0135	0.7440	0.2425	44.8290
2.4	0.0085	0.5965	0.3950	47.1140
2.5	0.0055	0.4735	0.5210	47.2740
2.6	0.0025	0.3535	0.6440	43.2020
2.7	0.0040	0.2390	0.7570	40.0830
2.8	0.0015	0.1840	0.8145	38.1430
2.9	0.0020	0.1360	0.8620	33.5915
3.0	0.0010	0.0855	0.9135	30.4290
3.1	0.0010	0.0525	0.9465	27.8360
3.2	0.0000	0.0430	0.9570	26.0625
3.3	0.0000	0.0225	0.9775	23.9900

TABLE XII

ERROR PROBABILITIES AND ASN VALUES FOR PROPOSED TEST OF THREE VALUES FOR THE MEAN OF AN EXPONENTIAL DENSITY, BOUNDARIES EXTENDED (1.0, 2.0, 3.0)

MEAN	Probability of Accepting H_{-1}	H_0	H_1	ASN
0.7	1.0000	0.0000	0.0000	7.2380
0.8	0.9985	0.0015	0.0000	8.2950
0.9	0.9785	0.0215	0.0000	9.8715
1.0	0.9445	0.0555	0.0000	11.6535
1.1	0.8525	0.1475	0.0000	13.3385
1.2	0.7465	0.2535	0.0000	14.7975
1.3	0.6150	0.3850	0.0000	16.1160
1.4	0.4655	0.5345	0.0000	17.2845
1.5	0.3595	0.6400	0.0005	18.1030
1.6	0.2675	0.7305	0.0020	19.1320
1.7	0.1905	0.8060	0.0035	21.3545
1.8	0.1395	0.8530	0.0075	23.3260
1.9	0.0990	0.8805	0.0205	26.9120
2.0	0.0815	0.8775	0.0410	30.2805
2.1	0.0595	0.8570	0.0835	34.3450
2.2	0.0480	0.8100	0.1420	39.5405
2.3	0.0350	0.7155	0.2495	43.2275
2.4	0.0260	0.5745	0.3995	45.3150
2.5	0.0210	0.4590	0.5200	45.6810
2.6	0.0110	0.3455	0.6435	41.7715
2.7	0.0150	0.2310	0.7540	38.6280
2.8	0.0075	0.1805	0.8120	36.1905
2.9	0.0095	0.1310	0.8595	31.7360
3.0	0.0075	0.0825	0.9100	28.2690
3.1	0.0060	0.0510	0.9430	25.5740
3.2	0.0040	0.0400	0.9560	23.9055
3.3	0.0030	0.0215	0.9755	21.3450

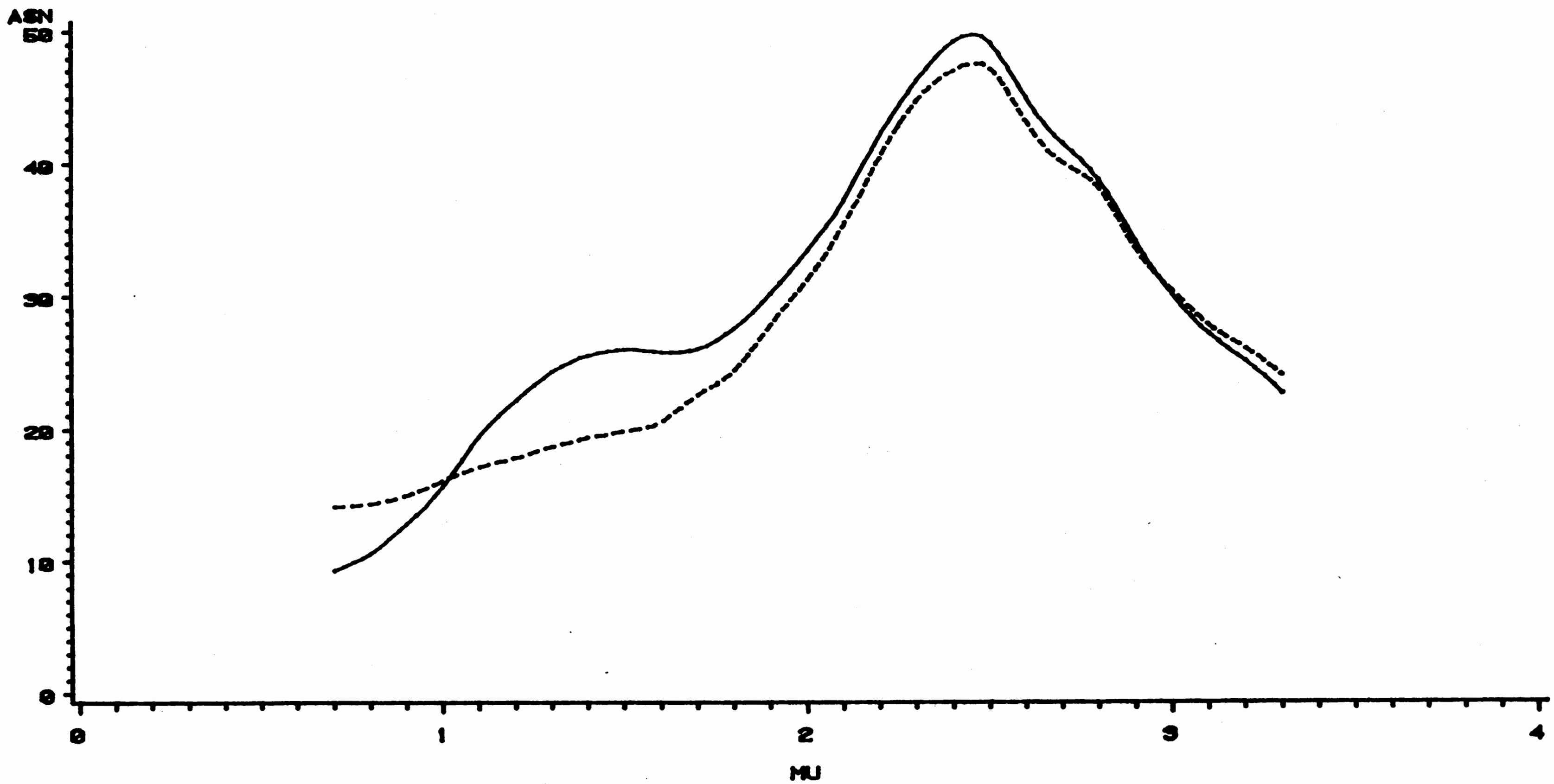

Figure 17. ASN Functions for Proposed Method (---) and Armitage's (——) for the Test of Three Exponential Means (1.0, 2.0, 3.0)

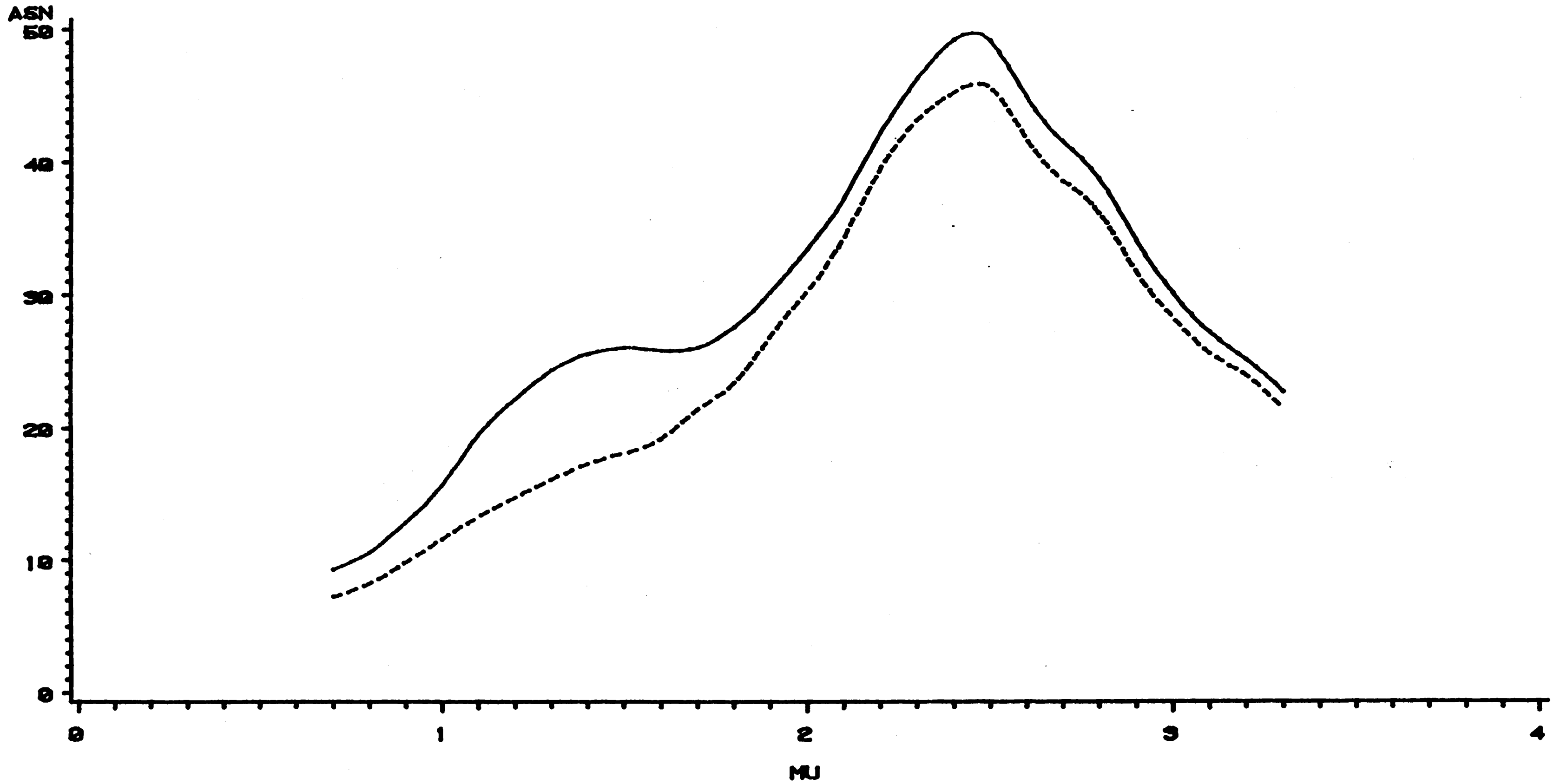

Figure 18. ASN Functions for Proposed Method (---) with Boundaries Extended and Armitage's (——) for Test of Three Exponential Means

Test for Normal Mean

Let x_1, x_2,..., x_n, ... be random observations from a normal distribution with unit variance and mean μ; i.e.

$$\begin{aligned} f(x) &= (2\pi)^{-1/2} \exp\{-(x - \mu)^2/2\} \\ &= \exp\{-\log (2\pi)/2 - x^2/2 + x\mu - \mu^2/2\}. \end{aligned} \tag{5.14}$$

Therefore, using the definition of Koopman-Darmois densities, $\theta = \mu$, $b(\theta) = \theta^2/2$, and $b'(\theta) = \theta$.

Suppose it is desired to test

$$H_{-1}: \ \mu = -1,$$
$$H_0 : \ \mu = 0,$$
and
$$H_1 : \ \mu = 1. \tag{5.15}$$

Assume the desired error probability is 0.05. The distribution of X_n will be normal with mean $n\mu$, variance n. h_0 and h_0' are the solutions to

$$h_0 = 2\theta - \theta_1 - \theta_0 = 2\theta - 1$$
$$h_0' = 2\theta - \theta_0 - \theta_{-1} = 2\theta + 1$$

When $\theta = \theta_{-1} = -1$, $h_0 = -3$ and $h_0' = 1$. When $\theta = \theta_0 = 0$, $h_0 = -1$ and $h_0' = 1$. When $\theta = \theta_1$, $h_0 = 1$ and $h_0' = 3$. The slopes of the upper and lower sets of parallel lines are found by

$$\text{slope}_1 = \frac{b(\theta_1) - b(\theta_0)}{\theta_1 - \theta_0} = 1/2$$

and

$$\text{slope}_0 = \frac{b(\theta_0) - b(\theta_{-1})}{\theta_0 - \theta_{-1}} = -1/2, \tag{5.16}$$

respectively.

As in the exponential case, the values a, c, d, and n_0 must be found in order to attain the approximate desired error probability of 0.05. Using Armitage's (1947) method, the values of a, c, d, and n_0 would be

$$a = 5.14,$$

$$c = 0,$$

$$d = -5.14,$$

and

$$n_0 = 4.5. \tag{5.17}$$

These will again be used to provide starting values to obtain the adjusted quantities. The system of equations associated with (5.15) is

$$\begin{aligned}
&L_1(\mu_1) + L_{-1}(\mu_1) = 0.05 \\
&L_1(\mu_{-1}) + L_{-1}(\mu_{-1}) = 0.05 \\
&1 - \Phi[(c - n_0\mu_0)/n_0^{.5}] - L_1(\mu_0) = 0.025 \\
&\Phi[(c - n_0\mu_0)/n_0^{.5}] - L_{-1}(\mu_0) = 0.025,
\end{aligned} \tag{5.18}$$

where

$$\Phi(k) = \int_{-\infty}^{k} (2\pi)^{-.5} \exp\{-x^2/2\}dx. \tag{5.19}$$

The system (5.18) then becomes

$$\begin{aligned}
&\exp\{-a\}\{\Phi[(a - n_0)/n_0^{.5}] - \Phi[(c - n_0)/n_0^{.5}]\} \\
&\div (\exp\{-a\} - \exp\{-c\}) - \exp\{-n_0/2\}\{\Phi[a/n_0^{.5}] \\
&- \Phi[c/n_0^{.5}]\}/(\exp\{-a\} - \exp\{-c\}) + \exp\{3n_0/2\} \\
&\times \{\Phi[(c + 2n_0)/n_0^{.5}] - \Phi[(d + 2n_0)/n_0^{.5}]\} \\
&\div (\exp\{-3c\} - \exp\{-3d\}) - \exp\{-3d\}\{\Phi[(c - n_0)/n_0^{.5}] \\
&- \Phi[(d - n_0)/n_0^{.5}]\}/(\exp\{-3c\} - \exp\{-3d\}) = 0.05
\end{aligned}$$

$$\begin{aligned}
&\exp\{3a\}\{\Phi[(a + n_0)/n_0^{.5}] - \Phi[(c - n_0)/n_0^{.5}]\} \\
&\div (\exp\{3a\} - \exp\{3c\}) - \exp\{3n_0/2\}\{\Phi[(a - 2n_0)/n_0^{.5}] \\
&- \Phi[(c - 2n_0)/n_0^{.5}]\}/(\exp\{3a\} - \exp\{3c\}) \\
&+ \exp\{-n_0/2\}\{\Phi[c/n_0^{.5}] - \Phi[d/n_0^{.5}]\}/(\exp\{c\} \\
&- \exp\{d\}) - \exp\{d\}\{\Phi[(c + n_0)n_0^{.5}] \\
&- \Phi[(d + n_0)/n_0^{.5}]\}/(\exp\{c\} - \exp\{d\}) = 0.05
\end{aligned}$$

$$\begin{aligned}
&1 - \Phi[c/n_0^{.5}] - \exp\{a\}\{\Phi[a/n_0^{.5}] - \Phi[c/n_0^{.5}]\} \\
&\div (\exp\{a\} - \exp\{c\}) + \exp\{n_0/2\}\{\Phi[(a - n_0)/n_0^{.5}] \\
&- \Phi[(c - n_0)/n_0^{.5}]\}/(\exp\{a\} - \exp\{c\}) = 0.025
\end{aligned}$$

$$\begin{aligned}
&\Phi[c/n_0^{.5}] - \exp\{n_0/2\}\{\Phi[(c + n_0)/n_0^{.5}] \\
&- \Phi[(d + n_0)/n_0^{.5}]\}/(\exp\{-c\} - \exp\{-d\}) \\
&+ \exp\{-d\}\{\Phi[c/n_0^{.5}] - \Phi[d/n_0^{.5}]\}/(\exp\{-c\} \\
&- \exp\{-d\}) = 0.025
\end{aligned} \tag{5.20}$$

The approximate solution given by PC SAS is

$$a = 5.89,$$

$$c = 0.0002,$$

$$d = -5.89,$$

and $$n_0 = 5.063. \qquad (5.21)$$

It will be of interest to compare this new method not only to Armitage's (1947) method, but to Billard and Vagholkar's as well. Their paper from 1969 lists several solutions to tests like the one performed here. For comparison, the test using $\alpha = \beta = 0.05$ and minimizing E(N) at $\mu = 0.5$ was used. The slope of this process was 0.4942, with

$$a = 5.975,$$

$$b = 0.1362,$$

$$c = -b,$$

$$d = -a,$$

and $$n_0 = 5.2358.$$

Figure 5 should be referred to for information on the appearance of Billard and Vagholkar's test.

Tables XIII through XVI give the results of Monte Carlo simulation designed to test the relative merits of each test. Note that the proposed method is a definite improvement over Armitage's. More importantly, the

TABLE XIII

ERROR PROBABILITIES AND ASN VALUES FOR ARMITAGE'S TEST OF THREE VALUES FOR THE MEAN OF A NORMAL DENSITY (-1.0, 0.0, 1.0)

MEAN	Probability of Accepting H_{-1}	H_0	H_1	ASN
-1.4	1.0000	0.0000	0.0000	4.9540
-1.3	0.9975	0.0025	0.0000	5.5535
-1.2	0.9985	0.0015	0.0000	6.3185
-1.1	0.9920	0.0080	0.0000	7.0065
-1.0	0.9795	0.0205	0.0000	8.3865
-0.9	0.9535	0.0465	0.0000	9.8795
-0.8	0.9020	0.0980	0.0000	11.9535
-0.7	0.8215	0.1785	0.0000	13.7635
-0.6	0.6545	0.3455	0.0000	15.7035
-0.5	0.4630	0.5365	0.0005	16.6500
-0.4	0.2620	0.7370	0.0010	15.3345
-0.3	0.1475	0.8505	0.0020	14.0650
-0.2	0.0710	0.9265	0.0025	12.3455
-0.1	0.0325	0.9595	0.0080	11.4085
0.0	0.0125	0.9700	0.0175	10.9275
0.1	0.0065	0.9590	0.0345	11.4620
0.2	0.0025	0.9305	0.0670	12.3530
0.3	0.0030	0.8445	0.1525	14.0115
0.4	0.0010	0.7035	0.2955	15.6190
0.5	0.0010	0.5410	0.4580	16.8260
0.6	0.0000	0.3470	0.6530	15.7045
0.7	0.0000	0.1890	0.8110	13.7240
0.8	0.0000	0.0940	0.9060	11.6930
0.9	0.0000	0.0435	0.9565	9.9145
1.0	0.0000	0.0160	0.9840	8.6105
1.1	0.0000	0.0085	0.9915	7.3305
1.2	0.0000	0.0050	0.9950	6.3290
1.3	0.0000	0.0010	0.9990	5.5080
1.4	0.0000	0.0000	1.0000	4.9730

TABLE XIV

ERROR PROBABILITIES AND ASN VALUES FOR BILLARD AND VAGHOLKAR'S TEST OF THREE VALUES FOR THE MEAN OF A NORMAL DENSITY (-1.0, 0.0, 1.0)

MEAN	Probability of Accepting H_{-1}	H_0	H_1	ASN
-1.4	0.9985	0.0015	0.0000	4.6420
-1.3	0.9955	0.0045	0.0000	5.1775
-1.2	0.9955	0.0045	0.0000	5.8695
-1.1	0.9880	0.0120	0.0000	6.5120
-1.0	0.9680	0.0320	0.0000	7.6320
-0.9	0.9340	0.0660	0.0000	8.8495
-0.8	0.8765	0.1235	0.0000	10.4540
-0.7	0.7905	0.2095	0.0000	11.9015
-0.6	0.6255	0.3745	0.0000	13.0205
-0.5	0.4520	0.5470	0.0010	13.6895
-0.4	0.2720	0.7265	0.0015	12.8010
-0.3	0.1600	0.8370	0.0030	11.9180
-0.2	0.0860	0.9095	0.0045	10.6665
-0.1	0.0405	0.9510	0.0085	10.0465
0.0	0.0145	0.9645	0.0210	9.7020
0.1	0.0090	0.9485	0.0425	9.9900
0.2	0.0030	0.9145	0.0825	10.7195
0.3	0.0035	0.8330	0.1635	11.9355
0.4	0.0010	0.6925	0.3065	13.0600
0.5	0.0010	0.5470	0.4520	13.4170
0.6	0.0000	0.3730	0.6270	13.1155
0.7	0.0000	0.2145	0.7855	11.8150
0.8	0.0000	0.1235	0.8765	10.1795
0.9	0.0000	0.0650	0.9350	8.9440
1.0	0.0000	0.0275	0.9725	7.8845
1.1	0.0000	0.0145	0.9855	6.7670
1.2	0.0000	0.0085	0.9915	5.8945
1.3	0.0000	0.0025	0.9975	5.1235
1.4	0.0000	0.0005	0.9995	4.6650

TABLE XV

ERROR PROBABILITIES AND ASN VALUES FOR PROPOSED TEST OF THREE VALUES FOR THE MEAN OF A NORMAL DENSITY (-1.0, 0.0, 1.0)

MEAN	Probability of Accepting H_{-1}	H_0	H_1	ASN
-1.4	0.9985	0.0015	0.0000	6.4695
-1.3	0.9955	0.0045	0.0000	6.8100
-1.2	0.9960	0.0040	0.0000	7.2525
-1.1	0.9890	0.0110	0.0000	7.7865
-1.0	0.9685	0.0315	0.0000	8.6645
-0.9	0.9345	0.0655	0.0000	9.7580
-0.8	0.8750	0.1250	0.0000	11.3320
-0.7	0.7885	0.2115	0.0000	12.5985
-0.6	0.6200	0.3800	0.0000	13.8120
-0.5	0.4400	0.5595	0.0005	14.3165
-0.4	0.2615	0.7380	0.0005	13.2635
-0.3	0.1475	0.8510	0.0015	12.3765
-0.2	0.0785	0.9180	0.0035	10.8805
-0.1	0.0350	0.9580	0.0070	10.2785
0.0	0.0100	0.9720	0.0180	9.8820
0.1	0.0055	0.9595	0.0350	10.2055
0.2	0.0015	0.9280	0.0705	11.0040
0.3	0.0010	0.8480	0.1510	12.3150
0.4	0.0005	0.7035	0.2960	13.5360
0.5	0.0005	0.5575	0.4420	14.0705
0.6	0.0000	0.3785	0.6215	13.8940
0.7	0.0000	0.2155	0.7845	12.5610
0.8	0.0000	0.1260	0.8740	10.9990
0.9	0.0000	0.0650	0.9350	9.8145
1.0	0.0000	0.0255	0.9745	8.9315
1.1	0.0000	0.0140	0.9860	7.9705
1.2	0.0000	0.0080	0.9920	7.2795
1.3	0.0000	0.0025	0.9975	6.7935
1.4	0.0000	0.0005	0.9995	6.4860

TABLE XVI

ERROR PROBABILITIES AND ASN VALUES FOR PROPOSED TEST OF THREE VALUES FOR THE MEAN OF A NORMAL DENSITY, BOUNDARIES EXTENDED (-1.0, 0.0, 1.0)

MEAN	Probability of Accepting H_{-1}	H_0	H_1	ASN
-1.4	0.9985	0.0015	0.0000	4.6410
-1.3	0.9955	0.0045	0.0000	5.1850
-1.2	0.9960	0.0040	0.0000	5.8915
-1.1	0.9890	0.0110	0.0000	6.5500
-1.0	0.9685	0.0315	0.0000	7.6685
-0.9	0.9355	0.0645	0.0000	8.9295
-0.8	0.8770	0.1230	0.0000	10.5925
-0.7	0.7905	0.2095	0.0000	12.0390
-0.6	0.6220	0.3780	0.0000	13.3410
-0.5	0.4485	0.5505	0.0010	13.8745
-0.4	0.2690	0.7295	0.0015	12.9580
-0.3	0.1585	0.8385	0.0030	12.0825
-0.2	0.0845	0.9105	0.0050	10.7440
-0.1	0.0385	0.9530	0.0085	10.1735
0.0	0.0140	0.9650	0.0210	9.7745
0.1	0.0085	0.9500	0.0415	10.0660
0.2	0.0030	0.9170	0.0800	10.8310
0.3	0.0035	0.8345	0.1620	12.0825
0.4	0.0010	0.6965	0.3025	13.2465
0.5	0.0010	0.5490	0.4500	13.6725
0.6	0.0000	0.3735	0.6265	13.3225
0.7	0.0000	0.2145	0.7855	11.9680
0.8	0.0000	0.1230	0.8770	10.2690
0.9	0.0000	0.0650	0.9350	8.9815
1.0	0.0000	0.0255	0.9745	7.9510
1.1	0.0000	0.0140	0.9860	6.7865
1.2	0.0000	0.0080	0.9920	5.9035
1.3	0.0000	0.0025	0.9975	5.1295
1.4	0.0000	0.0005	0.9995	4.6690

proposed method gives results very much like Billard and Vagholkar's. It should not give more favorable results for intermediate values of θ since the new method is a special case of Billard and Vagholkar's with b = c (Figure 5), and a restriction placed on the slopes. One main distinction, however, is that, if no minimum sample size is used, the boundaries for sampling in the proposed method extend to the X_n-axis. Thus for $\theta > \theta_1$ or for $\theta < \theta_{-1}$, the ASN for the new method should be smaller than that of Billard and Vagholkar.

In summary, the proposed procedure and Billard and Vagholkar's method give similar results. As explained in Chapter IV, however, their respective approaches differ. Billard and Vagholkar claim their method is optimal since it utilizes a minimization procedure on the ASN function. The method proposed in this dissertation should, therefore, also be optimal. See Figure 19 for a graphical representation of the proposed method (with the boundaries extended) and Figures 20 through 23 for comparisons of the proposed methods to that of Armitage's and Billard and Vagholkar's.

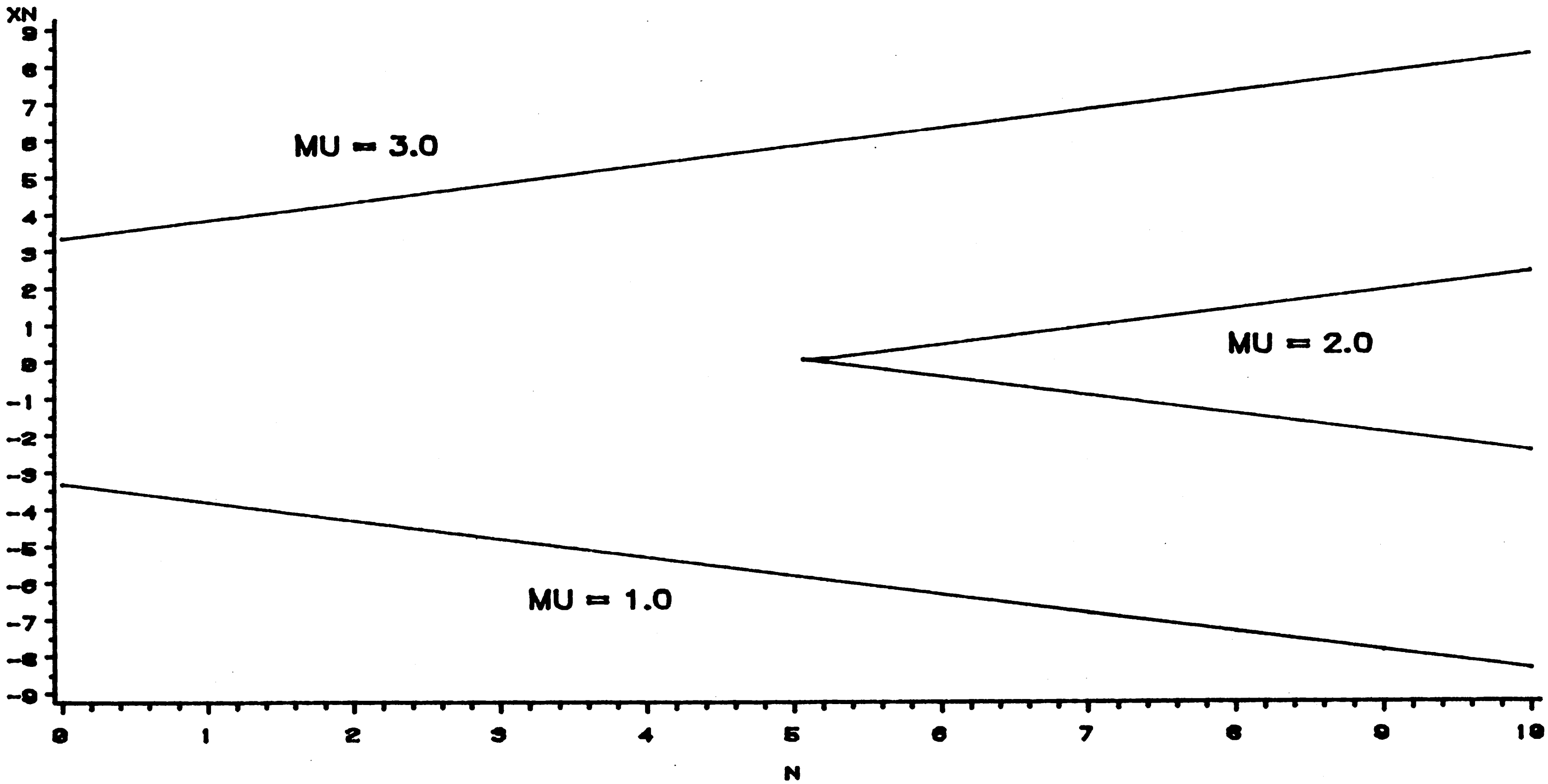

Figure 19. Sampling Region for Testing Three Values for the Mean of Standard Normal Distribution (-1.0, 0.0, 1.0), Boundaries Extended

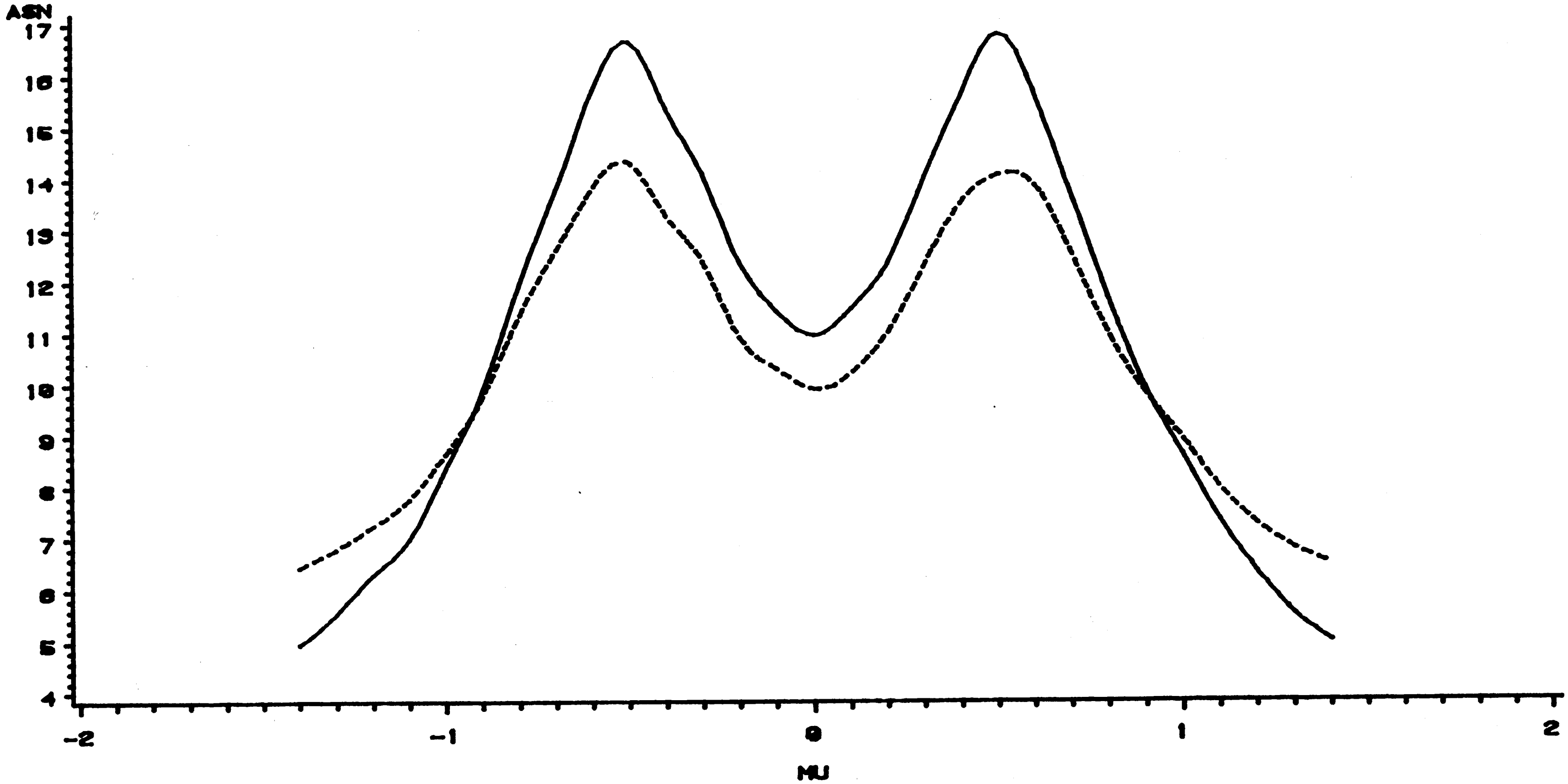

Figure 20. ASN Functions for Proposed Method (---) and Armitage's (——) for the Test of Three Standard Normal Means (-1.0, 0.0, 1.0)

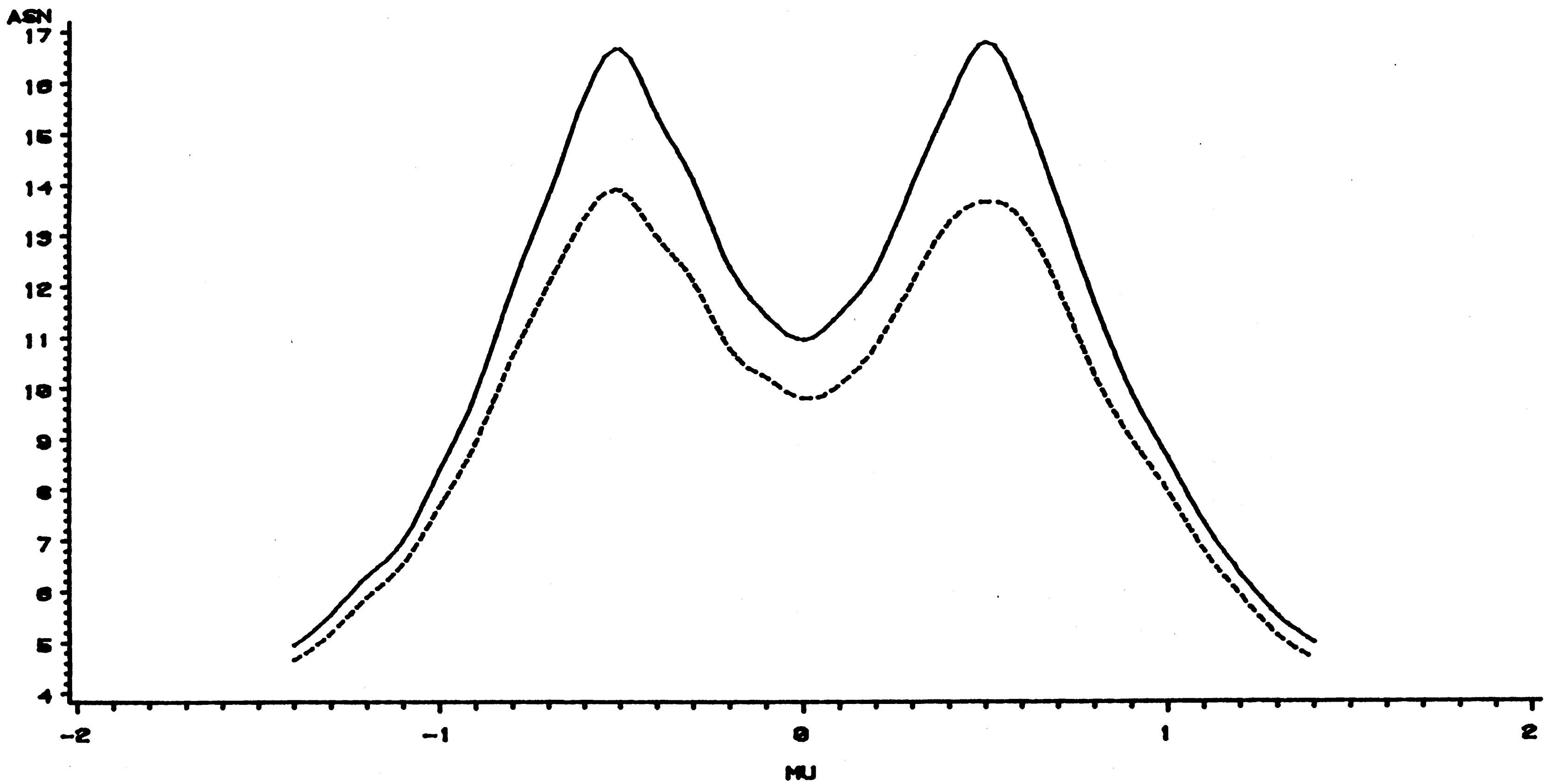

Figure 21. ASN Functions for Proposed Method (---) with Boundaries Extended and Armitage's (——) for Test of Three Normal Means

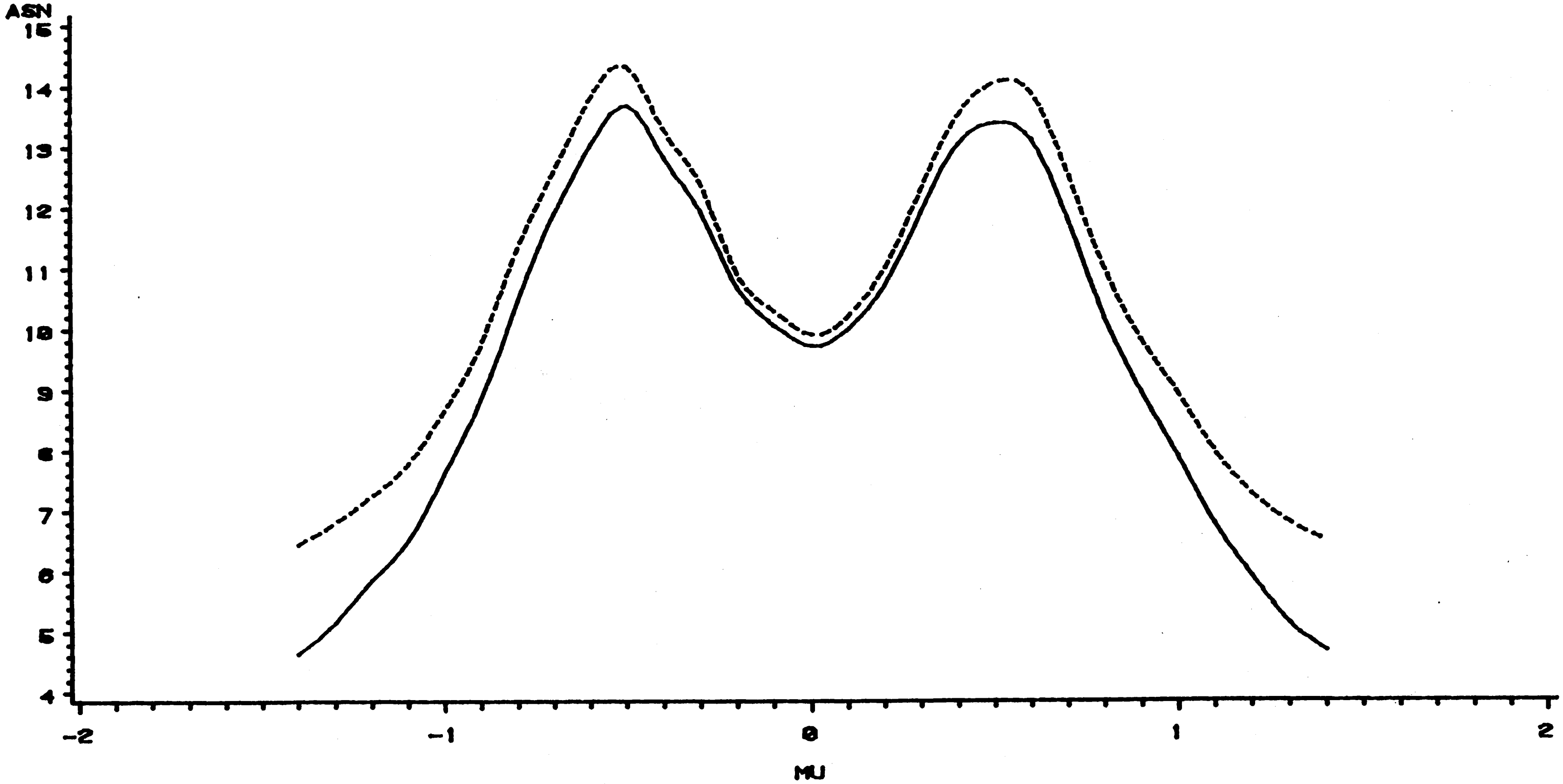

Figure 22. ASN Functions for Proposed Method (---) and Billard and Vagholkar's (——) for the Test of Three Standard Normal Means

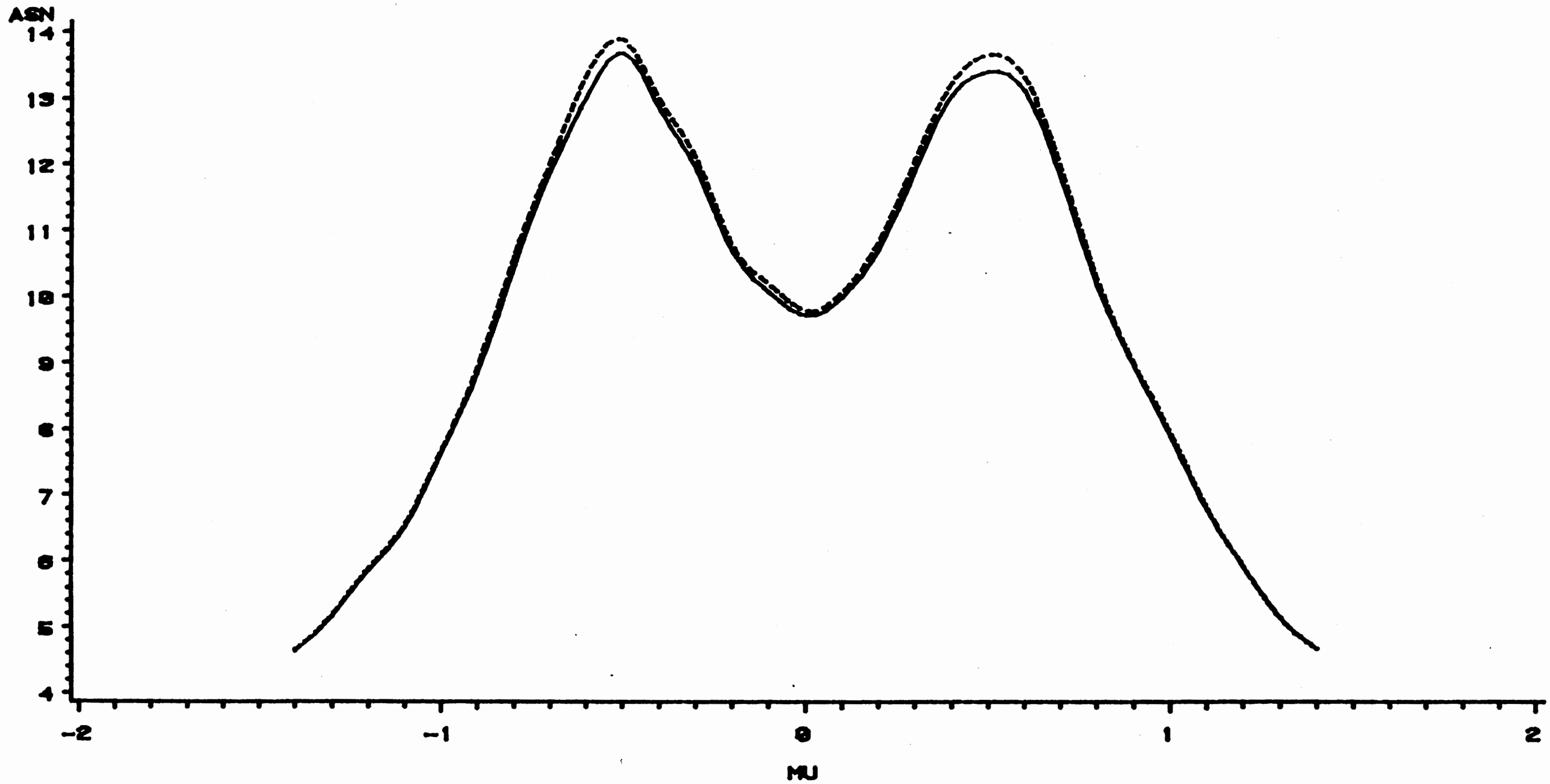

Figure 23. ASN Functions for Proposed Method (---) with the Boundaries Extended and Billard and Vagholkar's (——) for Normal Means

CHAPTER VI

A CLOSED PROCEDURE TO TEST THREE HYPOTHESES

Huffman (1983) developed a procedure, described in Chapter II, to sequentially test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$. His method, an extension of Lorden's 2-SPRT, provides an asymptotic solution to the Kiefer-Weiss problem. It involves determining the combination of one-sided SPRT's that will minimize the ASN function for parameter values between the hypothesized values. The continuation region is a closed triangular region depicted in Figure 3. A possible extension of the 2-SPRT to test a set of three hypotheses would be to simultaneously conduct two 2-SPRTs. One would decide between H_{-1} and H_0. The other would test H_0 vs. H_1.

The main focus of Huffman's work for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ with error rates α and β was determining θ^*, the intermediate value of θ that would result in the minimization of the ASN over the parameter space. Thus θ^* is determined so that a one-sided SPRT ($H_0 : \theta = \theta_0$ vs. $H_2 : \theta = \theta^*$) can be performed for the possible acceptance of H_0. Simultaneously, another one-sided SPRT ($H_2 : \theta = \theta^*$ vs. $H_1 : \theta = \theta_1$) is performed for the possible acceptance of H_1. An important aspect of Huffman's process is that he adjusts the error rates ($\alpha(\theta^*)$ and $\beta(\theta^*)$ of equations (2.16)) of the two one-sided tests so that the actual error rates are approximately equal to the nominal ones.

Consider again the problem of deciding among three hypotheses, but this time a closed test is desired. One possible approach is to determine θ_0^*, θ_1^*, α_0^*, α_1^*, β_0^*, and β_1^* such that two 2-SPRTs conducted simultaneously will yield predetermined error rates (see Figure 17). The procedure presented in Chapter IV of this thesis adjusted the original error rates of the two open SPRTs to attain desired probabilities of misclassification. This was accomplished by solving a system of four equations in four unknowns (a, c, d, and n_0), which are, in turn, functions of α_0, α_1, β_0, and β_1. The adjusted error rates used to construct the individual tests, α_0', α_1', β_0', and β_1', can be determined from a, c, d, and n_0. If these values are then used to construct the two individual 2-SPRTs as in Huffman's process, it would be quite natural to think that his adjustments (α_0^*, α_1^*, β_0^*, β_1^*), based upon α_0', α_1', β_0', β_1', would lend a closed three-hypothesis test with the desired error rates of α_0, α_1, β_0, and β_1. A Monte Carlo study was conducted to study the feasibility of this approach.

A Closed Test for the Exponential Parameter

Consider again the problem discussed in Chapter V of deciding among the hypotheses

$H_{-1}: \theta = -1,$

$H_0 : \theta = -1/2,$

and $H_1 : \theta = -1/3,$

where θ is the parameter from an exponential density. In Chapter V, it was determined that the approximate values of

$a = 50.41,$

$c = 20.94,$

$d = 14.89,$

and $n_0 = 13.96$

resulted in the specified error rates of $\alpha_0 = 0.10$, $\beta_0 = 0.05$, $\alpha_1 = 0.05$, $\beta_1 = 0.10$. The adjusted error rates of α_0', α_1', β_0', and β_1' that correspond with the values a, c, d, and n_0 can be found using equations (4.6). Applying (4.6), the adjusted error rates for this test are

$\alpha_0' = 0.42335,$

$\beta_0' = 0.06194,$

$\alpha_1' = 0.05755,$

and $\beta_1' = 0.10764.$ (6.1)

Huffman's 2-SPRT can now be applied twice; once for $H_{-1} : \theta = -1$ vs. $H_0 : \theta = -1/2$ and again for $H_0 : \theta = -1/2$ vs. $H_1 : \theta = -1/3$. The goal ultimately is to find the equations of the lines that determine the sampling regions; i.e., a_{00}, a_{01}, a_{10}, a_{11}, b_{00}, b_{01}, b_{10}, and b_{11} such that

$$\text{if } X_n \geq a_{11} + b_{11}n, \text{ accept } H_1,$$

$$\text{if } a_{01} + b_{01}n \leq X_n \leq a_{10} + b_{10}n, \text{ accept } H_0,$$

or

$$\text{if } X_n \leq a_{00} + b_{00}n, \text{ accept } H_{-1}. \tag{6.2}$$

These values are defined as in equation (2.17):

$$\begin{aligned}
a_{00} &= \log[\beta_0(\theta_0^*)/(1 - \alpha_0(\theta_0^*))]/(\theta_0^* - \theta_{-1}) \\
a_{01} &= \log[(1 - \beta_0(\theta_0^*))/\alpha_0(\theta_0^*)]/(\theta_0 - \theta_0^*) \\
a_{10} &= \log[\beta_1(\theta_1^*)/(1 - \alpha_1(\theta_1^*))]/(\theta_1^* - \theta_0) \\
a_{11} &= \log[(1 - \beta_1(\theta_1^*))/\alpha_1(\theta_1^*)]/(\theta_1 - \theta_1^*) \\
b_{00} &= [b(\theta_0^*) - b(\theta_{-1})]/(\theta_0^* - \theta_{-1}) \\
b_{01} &= [b(\theta_0) - b(\theta_0^*)]/(\theta_0 - \theta_0^*) \\
b_{10} &= b(\theta_1^*) - b(\theta_0)]/(\theta_1^* - \theta_0) \\
b_{11} &= b(\theta_1) - b(\theta_1^*)]/(\theta_1 - \theta_1^*)
\end{aligned} \tag{6.3}$$

θ_0^* and θ_1^* are the intermediate parameter values of the individual 2-SPRTs. First, the intermediate parameter values θ_0^* and θ_1^* must be found. Then the adjusted error rates must be adjusted further. Thus it remains to find θ_0^*, θ_1^*, and $\alpha_0(\theta_0^*)$, $\alpha_1(\theta_1^*)$, $\beta_0(\theta_0^*)$, $\beta_1(\theta_1^*)$.

First determine θ_0' and θ_1' such that

$$n_0^* = \log(1/\alpha_0')/I_{-1}(\theta_0') = \log(1/\beta_0')/I_0(\theta_0')$$

and

$$n_1^* = \log(1/\alpha_1')/I_0(\theta_1') = \log(1/\beta_1')/I_1(\theta_1') \tag{6.4}$$

where

$$I_i(\theta) = (\theta - \theta_i)b'(\theta) - \{b(\theta) - b(\theta_i)\} \text{ for } i = -1,\ 0,\ 1$$

For this case,

$$I_{-1}(\theta) = \log(-\theta) - 1 - 1/\theta,$$

$$I_0(\theta) = \log(-\theta) + \log 2 - 1 - 1/2\theta,$$

and

$$I_1(\theta) = \log(-\theta) + \log 3 - 1 - 1/3\theta. \tag{6.5}$$

Iterative solutions to (6.4) can be found to be

$$\theta'_0 = -0.7946055$$

and

$$\theta'_1 = -0.4059375 \tag{6.6}$$

which implies

$$n^*_0 = 30.0791$$

and

$$n^*_1 = 122.4918. \tag{6.7}$$

Define

$$a_i(\theta) = (\theta - \theta_i)/I_i(\theta) \quad \text{for } i = -1,\ 0,\ 1. \tag{6.8}$$

Find r^*_0 and r^*_1 such that

$$\Phi(r_0^*) = a_0(\theta_0')/[a_0(\theta_0')) - a_{-1}(\theta_0')]$$

and $$\Phi(r_1^*) = a_1(\theta_1')/[a_1(\theta_1')) - a_0(\theta_1')], \qquad (6.9)$$

where Φ is the cumulative distribution function for the standard normal density. For this exponential example,

$$a_1(\theta_1') = -3.98909,$$

$$a_0(\theta_1') = 4.03566,$$

$$a_0(\theta_0') = -3.18559,$$

and $$a_{-1}(\theta_0') = 7.187502, \qquad (6.10)$$

which implies

$$r_0^* = -0.504$$

and $$r_1^* = -0.00725. \qquad (6.11)$$

The values of θ_0^* and θ_1^* can be found by

$$\theta_i^* = \theta_i' \frac{r_i^*}{\sigma_i^*(n_i^*)^{0.5}} \quad \text{for } i = 0, 1, \qquad (6.12)$$

where

$$\sigma_i^* = [\mathrm{Var}(X|\theta = \theta_i')]^{0.5}.$$

For this test,

$$\sigma_0^* = 1/\theta_0' = 1.25849$$

and $$\sigma_1^* = 1/\theta_1' = 2.46343. \qquad (6.13)$$

Employing equations (6.6), (6.7), (6.11), and (6.13), values for θ_0^* and θ_1^* are

$$\theta_0^* = -0.867626$$

and $$\theta_1^* = -0.406203. \qquad (6.14)$$

To find $\alpha_0(\theta_0^*)$, $\alpha_1(\theta_1^*)$, $\beta_0(\theta_0^*)$, and $\beta_1(\theta_1^*)$, equation (2.16) will be used. The formulae are

$$\alpha_i(\theta_i^*) = \frac{a_{i-1}(\theta_i^*) - a_i(\theta_i^*)}{a_{i-1}(\theta_i^*)}\,\alpha_i' \quad \text{for } i = 0,\ 1,$$

$$\beta_i(\theta_i^*) = \frac{a_i(\theta_i^*) - a_{i-1}(\theta_i^*)}{a_i(\theta_i^*)}\,\beta_i' \quad \text{for } i = 0,\ 1. \qquad (6.15)$$

a_i is defined in equation (6.8) and α_i and β_i in equations (6.1). Thus

$$\alpha_0(\theta_0^*) = 0.52092,$$
$$\beta_0(\theta_0^*) = 0.07622,$$
$$\alpha_1(\theta_1*) = 0.11408,$$
$$\beta_1(\theta_1^*) = 0.21724. \qquad (6.16)$$

All values needed to complete the computations of equations

(6.3) are now known. These are then used to determine the final form of the test as given in equations (6.2).

Therefore, the sampling scheme is to continue sampling until one of the following conditions is met:

1) Accept H_{-1} if $X_n \leq 1.4992n - 5.0003$.
2) Accept H_0 if $X_n \leq 2.7132n - 19.2894$ and $X_n \geq 1.0727n + 4.3278$.
3) Accept H_1 if $X_n \geq 2.2149n + 20.5331$.

Figure 24 is a graphical representation of the closed procedure developed in this chapter. Table XVII presents the results of a Monte Carlo simulation study comparing this closed procedure with the open procedure derived in Chapter V, with Figure 25 the corresponding graph comparing empirical Average Sample Number functions. The primary motivation for developing this test is the possible reduction of sample sizes at intermediate parameter values. In view of Table XVII, the goal appears to have been attained. The error rates that were previously observed with the open test of Chapter IV were not disturbed greatly by the closure of this procedure. While the form of this test is more difficult to derive, a SAS software package could make it easy for users to implement.

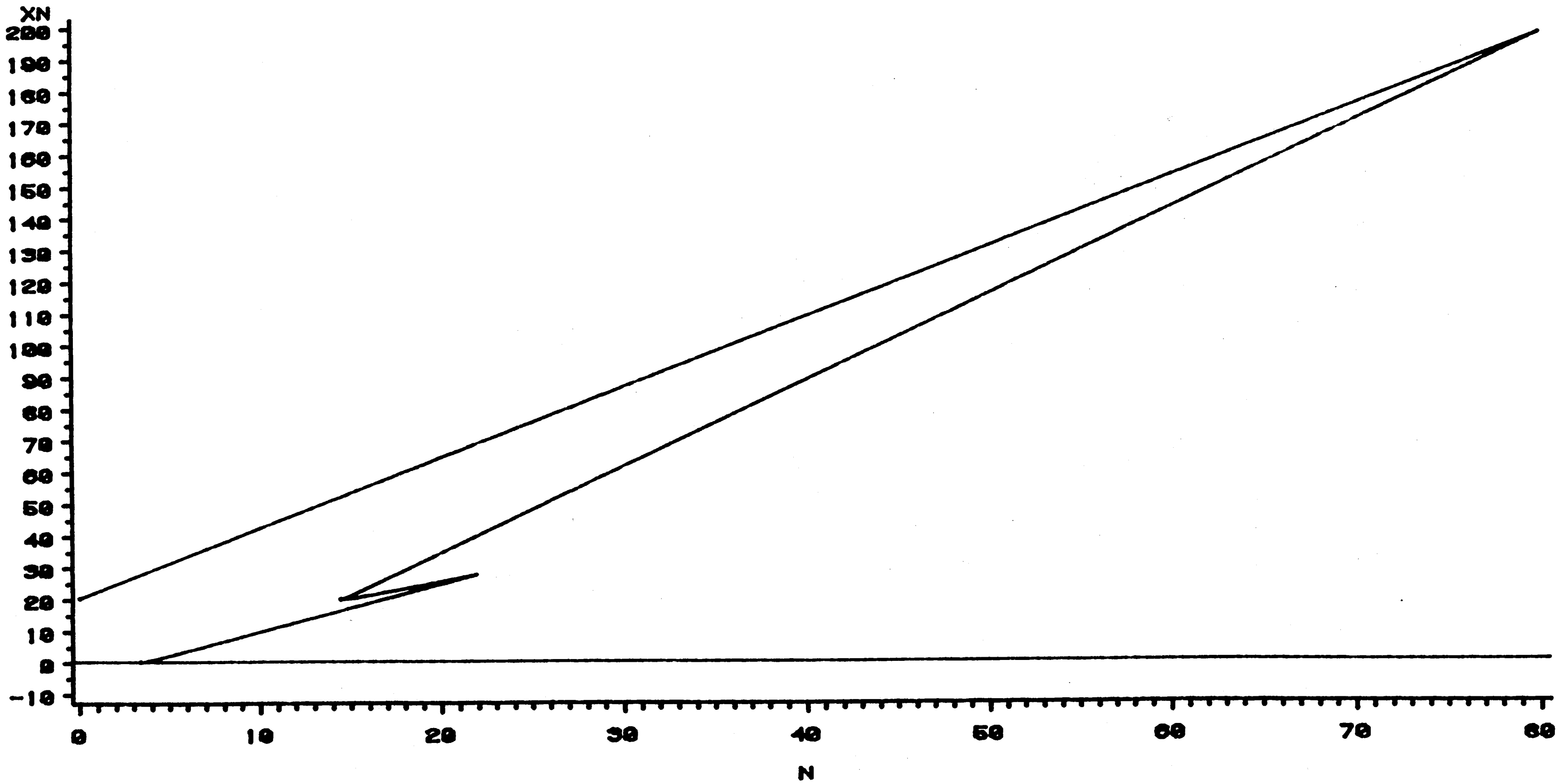

Figure 24. Closed Method for Testing Values for the Mean of an Exponential Distribution (1.0, 2.0, 3.0)

TABLE XVII

ERROR PROBABILITIES AND ASN VALUES FOR PROPOSED CLOSED TEST OF THREE VALUES FOR THE MEAN OF AN EXPONENTIAL DENSITY (1.0, 2.0, 3.0)

MEAN	Probability of Accepting H_{-1}	H_0	H_1	ASN
0.7	0.9985	0.0015	0.0000	10.1910
0.8	0.9945	0.0055	0.0000	10.7695
0.9	0.9580	0.0420	0.0000	11.5665
1.0	0.8825	0.1175	0.0000	12.3135
1.1	0.7730	0.2270	0.0000	13.2975
1.2	0.6685	0.3315	0.0000	14.1055
1.3	0.5435	0.4565	0.0000	15.1085
1.4	0.4200	0.5800	0.0000	16.2600
1.5	0.3290	0.6705	0.0005	17.2725
1.6	0.2355	0.7625	0.0020	18.5970
1.7	0.1775	0.8180	0.0045	20.5760
1.8	0.1350	0.8555	0.0095	22.1105
1.9	0.0945	0.8790	0.0265	24.6035
2.0	0.0690	0.8775	0.0535	26.5870
2.1	0.0510	0.8475	0.1015	28.4335
2.2	0.0390	0.7875	0.1735	30.3740
2.3	0.0270	0.7245	0.2485	32.0715
2.4	0.0185	0.5990	0.3825	32.7035
2.5	0.0130	0.4960	0.4910	32.4365
2.6	0.0065	0.4055	0.5880	31.4405
2.7	0.0095	0.3015	0.6890	30.0670
2.8	0.0045	0.2365	0.7590	28.9695
2.9	0.0035	0.1840	0.8125	27.6225
3.0	0.0030	0.1170	0.8800	25.1730
3.1	0.0030	0.0865	0.9105	23.4400
3.2	0.0000	0.0710	0.9290	22.3620
3.3	0.0020	0.0275	0.9705	20.6430

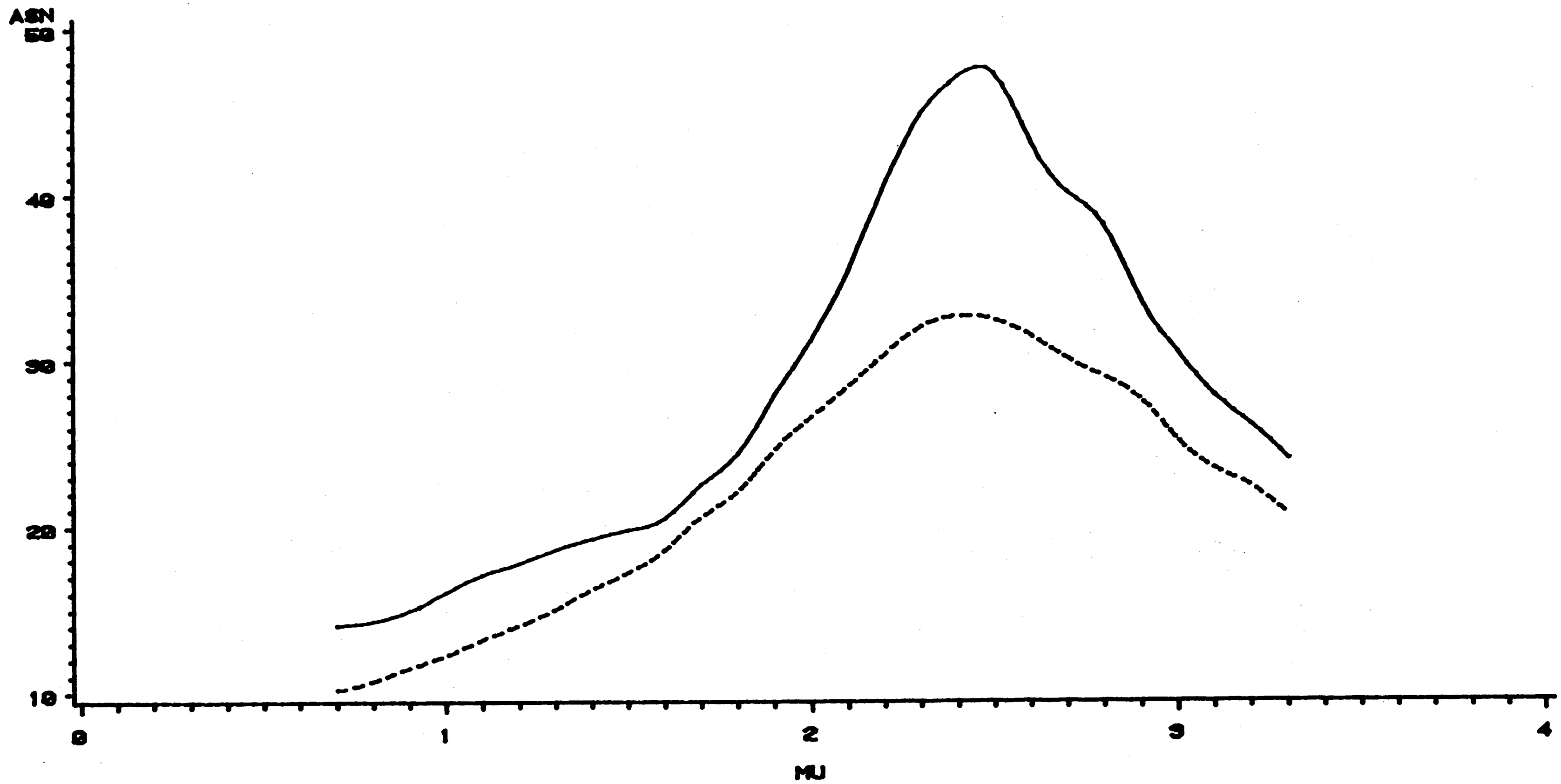

Figure 25. ASN Functions for Proposed Closed (---) and Open (——) Methods for Test of Three Exponential Means (1.0, 2.0, 3.0)

CHAPTER VII

SUMMARY AND CONCLUSIONS

Several sequential methods for testing simple versus simple hypotheses exist. Attempts have been made to extend this two-hypothesis case to a three-hypothesis case or, similarly, to a simple versus composite case. Often these proposed procedures involve simultaneously conducting two sequential tests of simple hypotheses. Problems arise, however, when application of one procedure to all testing situations is attempted.

Formulae exist that approximate the probability that a random walk starting at a certain point will cross one of two parallel boundaries. These formulae, developed by Billard and Vagholkar (1969), are used in this dissertation to approximate the probabilities of error in Armitage's (1947) method to sequentially decide among three hypotheses. These formulae are functions of the error rates used to determine sampling regions. The values these error rates must be set to in order that the procedure will approximately attain desired error probabilities are then established. This creates a non-linear system of four equations in four unknowns. Chapter IV developed the theory needed to determine these equations.

In Chapter V, a SAS program to solve any system of non-linear equations is presented, and it is used to solve the system developed in Chapter IV. Though this process was developed for any distribution in the Koopman-Darmois family, it is necessary for the distribution function either to have a closed form or to be a defined SAS function. This will permit the MODEL statement in PROC NLIN to include the cumulative distribution functions necessary to solve the system of four equations. In the rare event the test was needed for a Koopman-Darmois density not represented by a SAS function, other methods can be employed to estimate the corresponding cumulative distribution function. The fact that any Koopman-Darmois density can be tested by this method is an advantage. Billard and Vagholkar's procedure was derived for the normal mean and the binomial parameter only. However, with considerable effort, any Koopman-Darmois distribution could be derived using their method.

Examining the Monte Carlo simulations is one way of determining the merits of the proposed method versus Armitage's or Billard and Vagholkar's. For both the exponential and normal examples, the proposed method is an improvement over Armitage's in that the error rates are closer to the nominal levels and the ASN function is smaller. When the proposed method is compared to Billard and Vagholkar for the normal case, the two procedures give similar results. An advantage of the proposed method is that it may be implemented using readily available software.

This proposed method was used later to derive a closed sequential procedure to test three hypotheses. Huffman's extension of Lorden's work was used twice with the adjusted error rates obtained from the system of four equations. The example performed in Chapter VI was with the exponential distribution, mainly because the open procedure gave large ASN values for parameter values intermediate to the hypothesized ones. Therefore, a closed procedure would naturally be desired for this case. The closed procedure did reduce the ASN at intermediate parameter values as desired while maintaining the specified error rates.

The application of these methods to other distributions will be topics of future research. A natural candidate for this procedure is the binomial distribution, mainly due to applications in medicine with clinical trials and process sampling in industrial engineering. Another distribution that can be studied is the negative binomial with its application to entomology.

The procedure developed in this thesis works only for tests with three simple hypotheses. Extension to a test of more than three hypotheses would require development of a more complex system of equations. Instead of solving a system of four equations, four unknowns, a system of six or seven equations might be involved for a test of four hypotheses. Such an extension would be a possible topic of further research.

Further development of the closed procedure introduced in Chapter VI could also be considered for future research. The asymptotic distribution of this procedure will be of interest in order to obtain a test with desirable properties.

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V

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